L'ordre faible facial

et tout son gloire

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Outline

- How to arrange hyperplanes.
- The facial weak order in all its glory.
- The path of least resistance.
- What else?

How to arrange hyperplanes

A basic human problem

A basic human problem

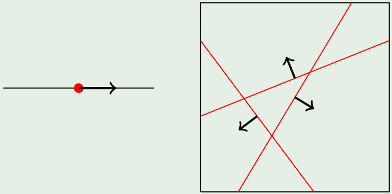
A basic human problem

A basic human problem

What is a hyperplane?

- $(V, \langle\cdot,\cdot\rangle)$ - n-dim real Euclidean vector space.
- A *hyperplane H* is $\operatorname{codim} 1$ subspace of V with normal e_H.

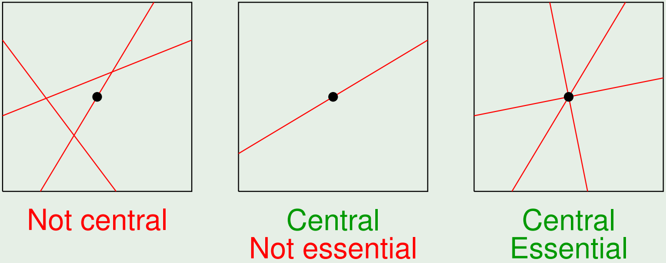
Example



Arranging hyperplanes

- A *hyperplane arrangement* is $\mathcal{A} = \{H_1, H_2, \ldots, H_k\}$.
- $\mathcal{A}$ is *central* if $\{0\} \subseteq \bigcap \mathcal{A}$.
- Central $\mathcal{A}$ is *essential* if $\{0\} = \bigcap \mathcal{A}$.

Example

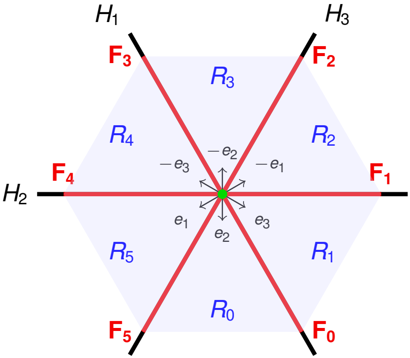


In terms of food?

Central essential hyperplane arrangement

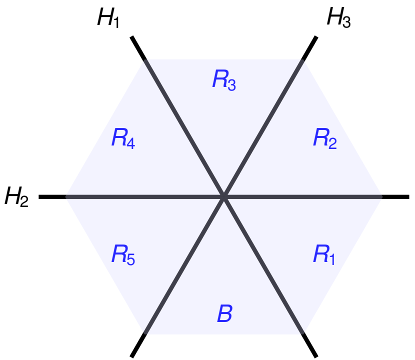
Exploding arrangements

- *Regions* $\mathscr{R}_\mathcal{A}$ - closures of connected components of V without $\mathcal{A}$.
- *Faces* $\mathscr{F}_\mathcal{A}$ - intersections of some regions.



A regional order

- *Base region* $B \in \mathscr{R}_{\mathcal{A}}$ - some fixed region
- *Separation set for* $R \in \mathscr{R}_{\mathcal{A}}$
  $S(R) := \{H \in \mathcal{A} \mid H \text{ separates } R \text{ from } B\}$



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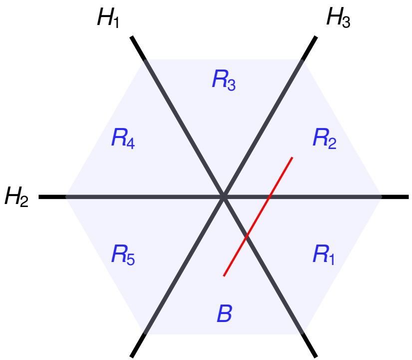

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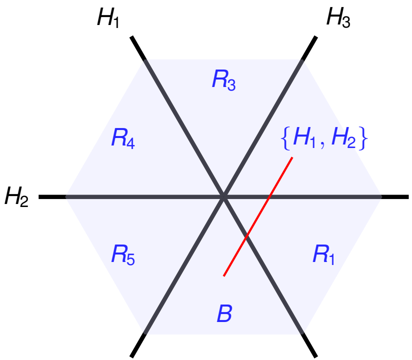

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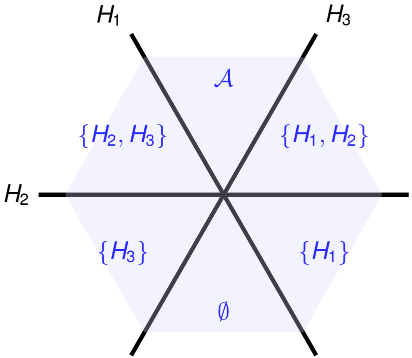

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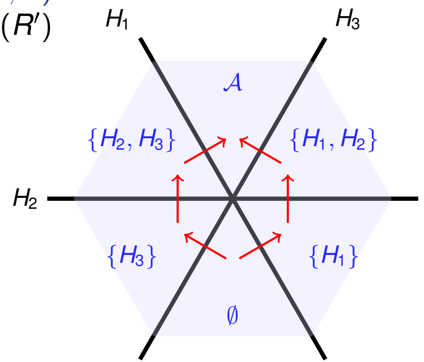
  $S(R) := \{H \in \mathcal{A} \mid H \text{ separates } R \text{ from } B\}$
- *Poset of Regions* $\mathrm{PR}(\mathcal{A}, B)$ where
  $R \leq_{\mathrm{PR}} R' \Leftrightarrow S(R) \subseteq S(R')$



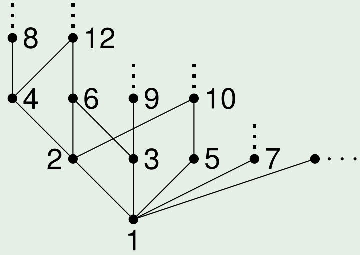
Ordering all the things

- *Lattice* - poset where every two elements have a *meet* (greatest lower bound) and *join* (least upper bound).

Example

The lattice $(\mathbb{N}, |)$ where $a \leq b \Leftrightarrow a \mid b$.

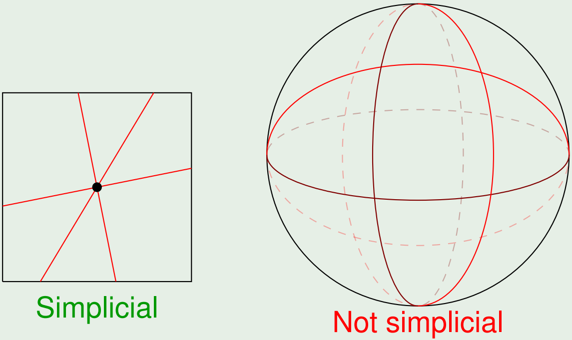
- meet - greatest common divisor
- join - least common multiple



Simply simplicial arrangements

- A region R is *simplicial* if normal vectors for boundary hyperplanes are linearly independent.
- $\mathcal{A}$ is *simplicial* if all $\mathscr{R}_{\mathcal{A}}$ simplicial.

Example

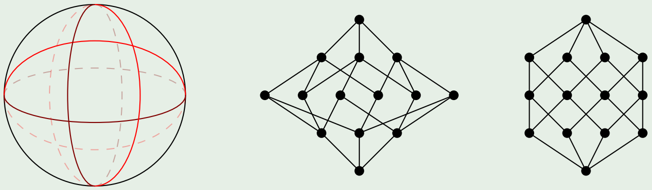


A regional lattice

Theorem (Björner, Edelman, Ziegler '90)

If $\mathcal{A}$ is simplicial then $\mathrm{PR}(\mathcal{A}, B)$ is a lattice for any $B \in \mathscr{R}_{\mathcal{A}}$. If $\mathrm{PR}(\mathcal{A}, B)$ is a lattice then B is simplicial.

Example



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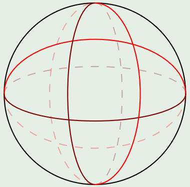
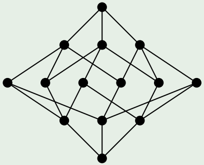
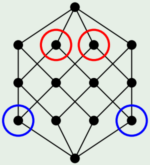




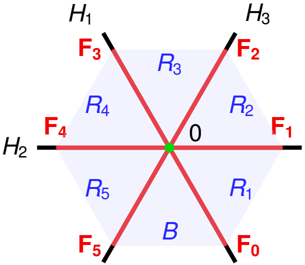


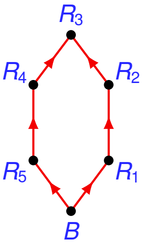
Facial weak order in all its glory

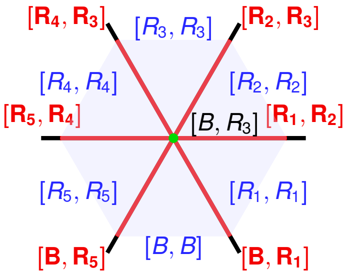
Facial intervals

Proposition (Björner, Las Vergas, Sturmfels, White, Ziegler '93)

Let $\mathcal{A}$ be central with base region B. For every $F \in \mathscr{F}_\mathcal{A}$ there is a unique interval $[m_F, M_F]$ in $\mathrm{PR}(\mathcal{A}, B)$ such that $[m_F, M_F] = \{R \in \mathscr{R}_\mathcal{A} \mid F \subseteq R\}$







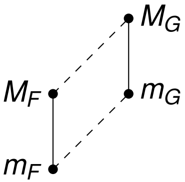
Facial weak order (!!!)

Let $\mathcal{A}$ be a central hyperplane arrangement and B a base region in $\mathscr{R}_{\mathcal{A}}$.

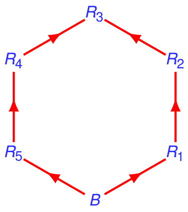
Definition

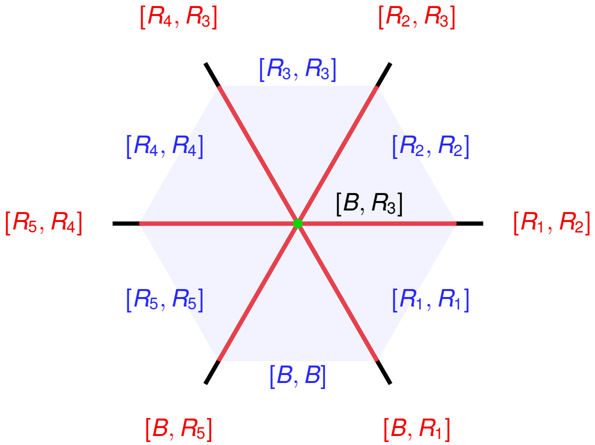
The *facial weak order* is the order $\mathrm{FW}(\mathcal{A}, B)$ on $\mathscr{F}_{\mathcal{A}}$ where for $F, G \in \mathscr{F}_{\mathcal{A}}$:

$$F \leq G \Leftrightarrow m_F \leq_{\mathrm{PR}} m_G \text{ and } M_F \leq_{\mathrm{PR}} M_G$$



A first example





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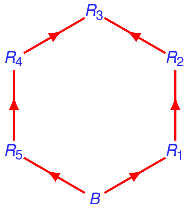

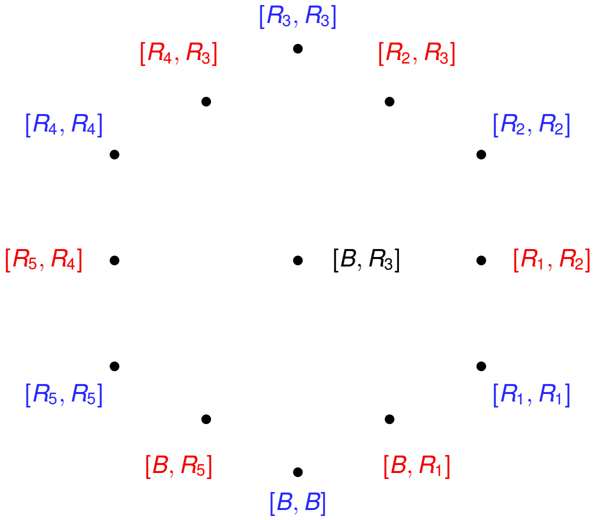

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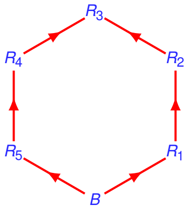

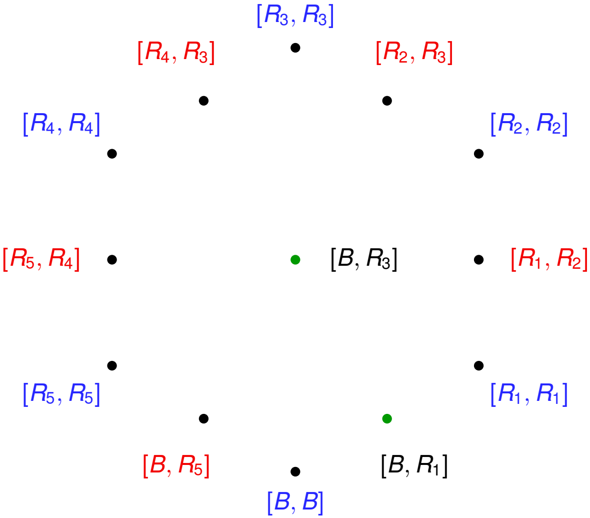

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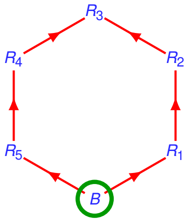

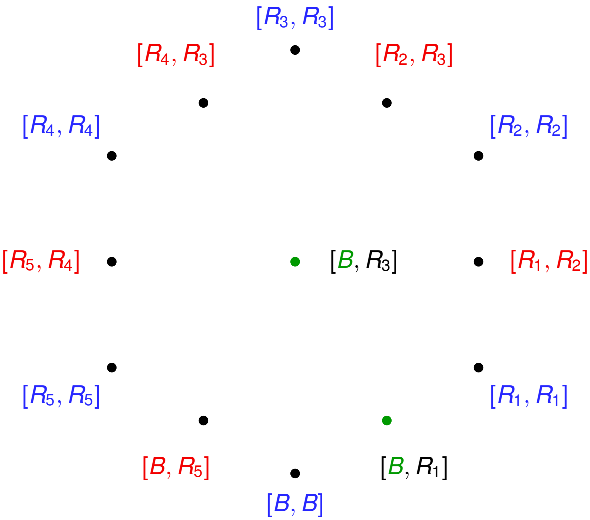

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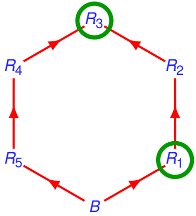

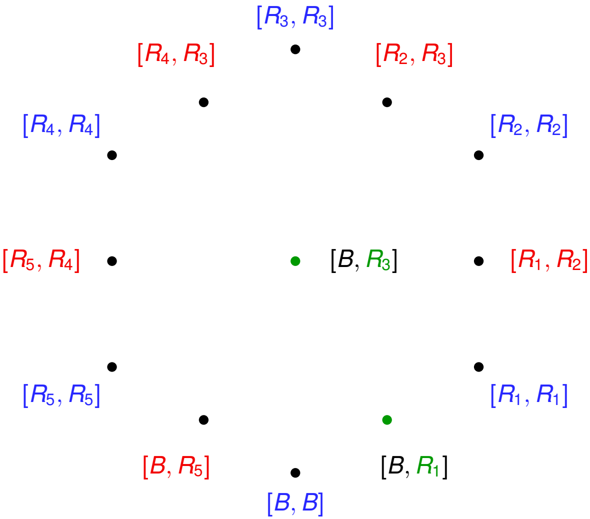

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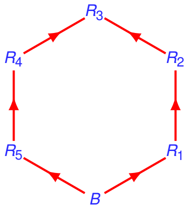

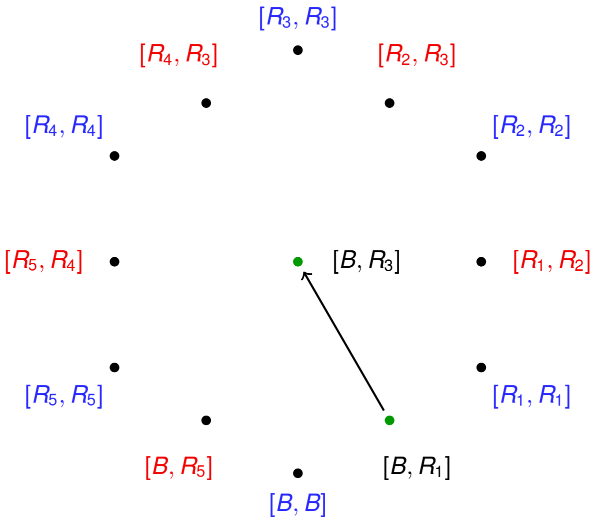

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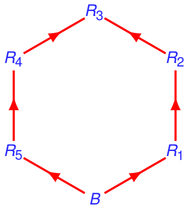
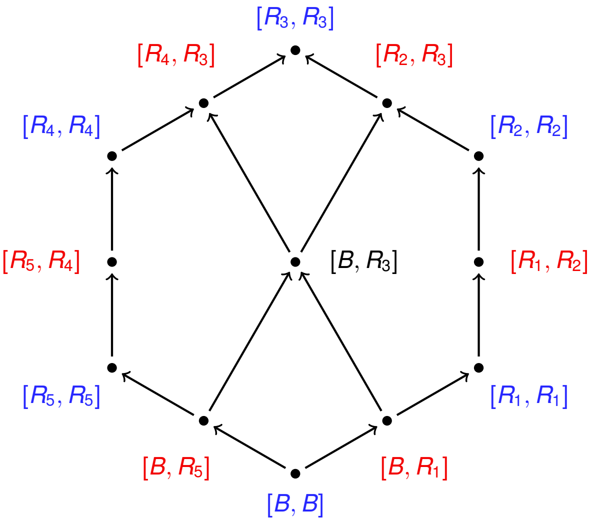




Facial weak order lattice

Theorem (D., Hohlweg, McConville, Pilaud '19+)

The facial weak order $\mathrm{FW}(\mathcal{A}, B)$ *is a lattice when* $\mathrm{PR}(\mathcal{A}, B)$ *is a lattice.*

Rewind: How did we get here?

The origins

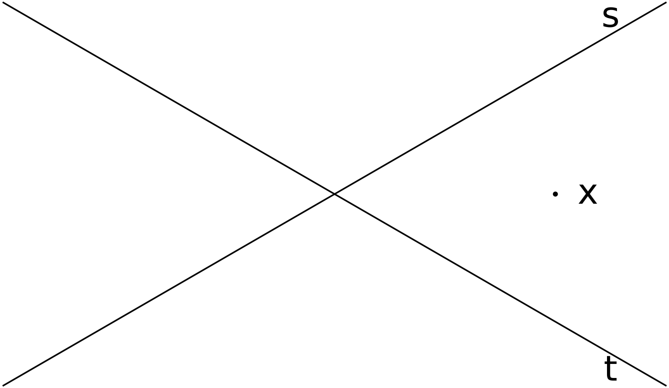
- **2001:** Krob, Latapy, Novelli, Phan, and Schwer extended the weak order of Coxeter groups to an order on all the faces of its associated arrangement for type A.
- **2006:** Palacios and Ronco extended this new order to Coxeter groups of all types using cover relations.

The origins

- **2001:** Krob, Latapy, Novelli, Phan, and Schwer extended the weak order of Coxeter groups to an order on all the faces of its associated arrangement for type A.
- **2006:** Palacios and Ronco extended this new order to Coxeter groups of all types using cover relations.
- Questions:
  - Can we extend this to all Coxeter group types and hyperplane arrangements?
  - Can we find both local and global definitions?
  - When do we actually get a lattice?

The infamous Coxeter

Coxeter's Idea



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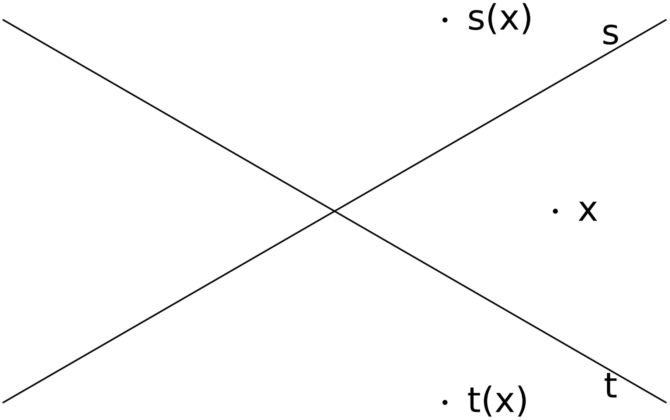

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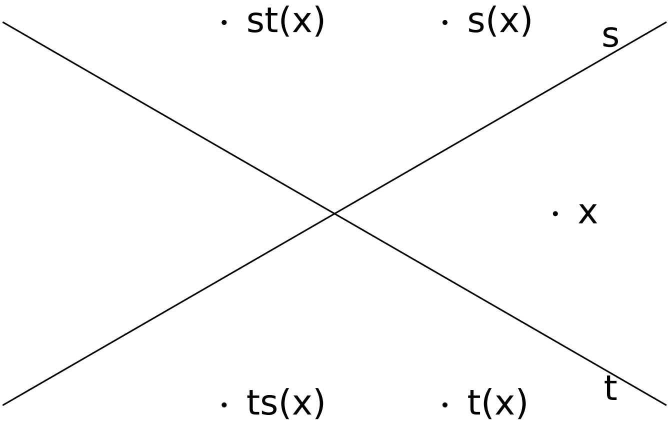

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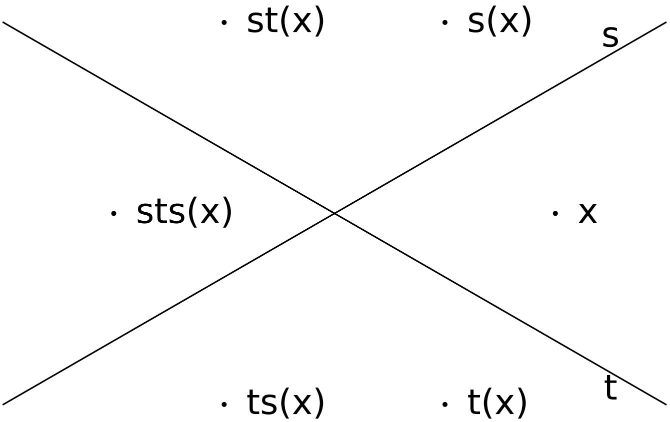
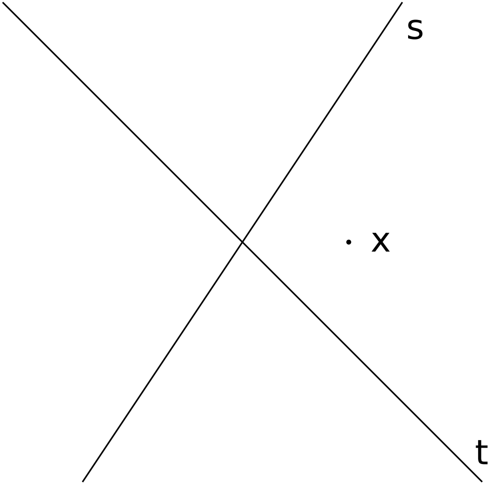


A failure



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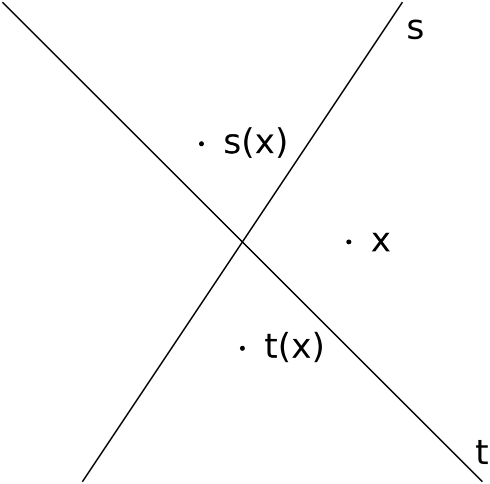

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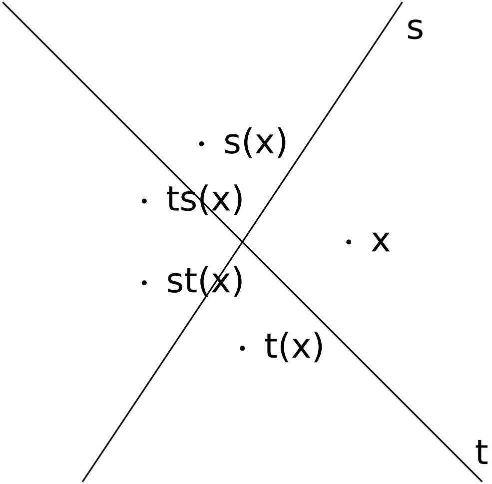

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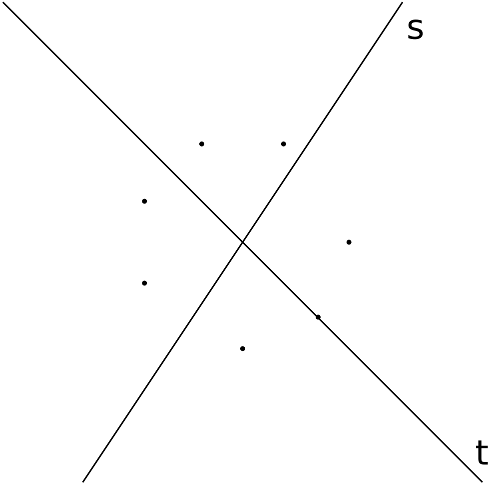

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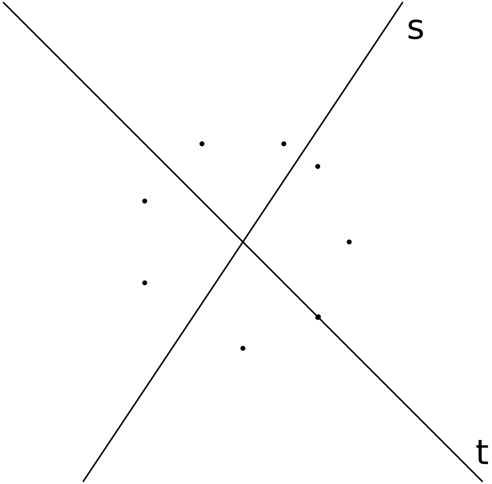

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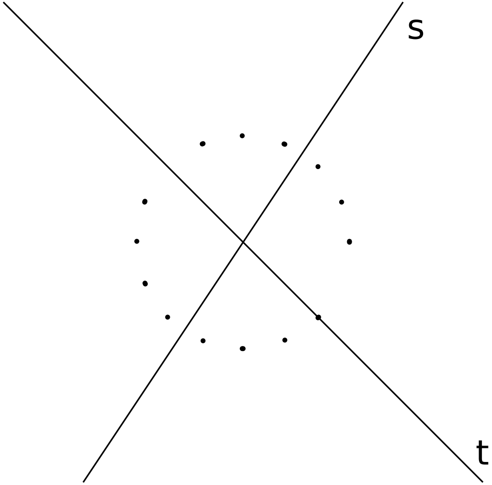

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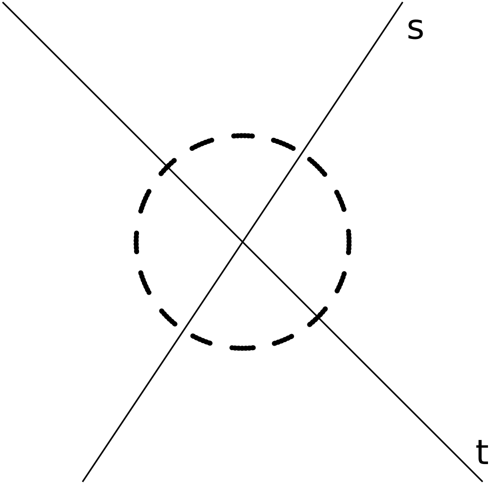

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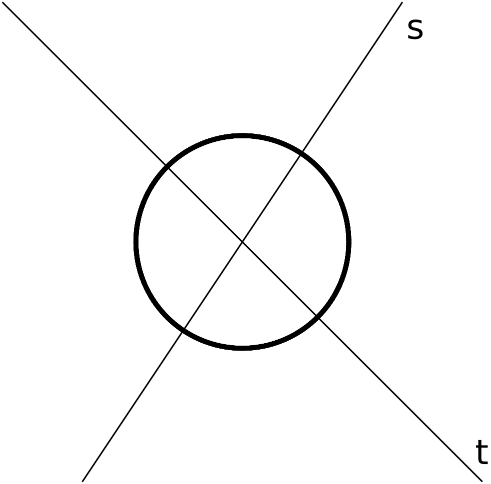


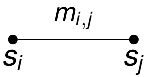
Coxeterian systems

- *Finite Coxeter System* (W, S) such that

$$W := \langle s \in S \mid (s_i s_j)^{m_{i,j}} = e \text{ for } s_i, s_j \in S \rangle$$

where $m_{i,j} \in \mathbb{N}^\star$ and $m_{i,j} = 1$ only if $i = j$.

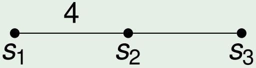
- A *Coxeter diagram* Γ_W for a Coxeter System (W, S) has S as a vertex set and an edge labelled $m_{i,j}$ when $m_{i,j} > 2$.



Example

$$W_{B_3} = \left\langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^4 = (s_2 s_3)^3 = (s_1 s_3)^2 = e \right\rangle$$

Γ_{B_3} :



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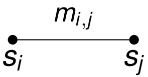
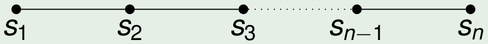


Example

$W_{A_n} = S_{n+1}$, symmetric group.

Γ_{A_n} :



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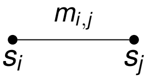
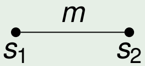


Example

$W_{I_2(m)} = \mathcal{D}(m)$, dihedral group of order $2m$.

$\Gamma_{I_2(m)}$:



A not so strong order

Let (W, S) be a Coxeter system.

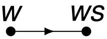
- Let $w \in W$ such that $w = s_1 \dots s_n$ for some $s_i \in S$. We say that w has *length* n, $\ell(w) = n$, if n is minimal.

Example

Let Γ_{A_2} : 

.

$\ell(stst) = 2$ as $stst = tstt = ts$.

- Let the *(right) weak order* be the order $\leq_R$ on the Cayley graph where 

and $\ell(w) < \ell(ws)$.

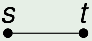
A not so strong lattice

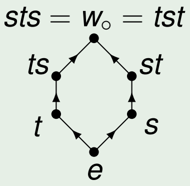
Theorem (Björner '84)

Let (W, S) be a finite Coxeter system. The weak order is a lattice graded by length.

- For finite Coxeter systems, there exists a longest element in the weak order, $w_\circ$.

Example

Let Γ_{A_2} : .

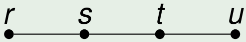


Parabolic Subgroups

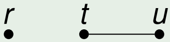
(W, S) a Coxeter system and $I \subseteq S$.

- $W_I = \langle I \rangle$ — *standard parabolic subgroup* (long elt: $w_{\circ,I}$).
- $W^I := \{w \in W \mid \ell(w) \leq \ell(ws)$, for all $s \in I\}$ is the set of min length coset representatives for W/W_I.
- Unique factorization: $w = w^I \cdot w_I$ with $w^I \in W^I$, $w_I \in W_I$.
- By convention in this talk xW_I means $x \in W^I$.

Example

Let Γ_W : 

and $I = \{r, t, u\}$.

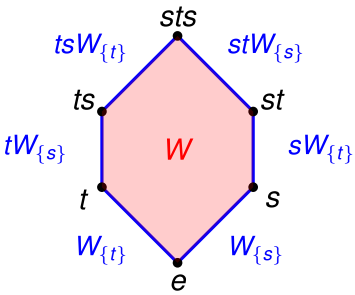
Then Γ_{W_I} : 

$w = rtustr \qquad w = rts \cdot utr$

So complex

(W, S) a Coxeter system and $I \subseteq S$.

- *Coxeter complex* - $\mathcal{P}_W$ - complex whose faces are all the standard parabolic cosets of W.



The first stepping stone

Let (W, S) be a finite Coxeter system.

Definition (Krob et.al. '01, type A; Palacios, Ronco '06)

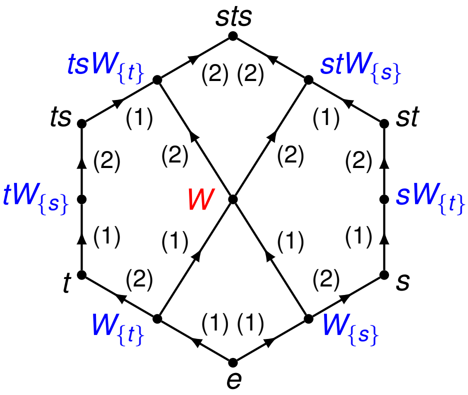
The *(right) facial weak order* is the order $\leq_F$ on the Coxeter complex $\mathcal{P}_W$ defined by cover relations of two types:

$$
\begin{aligned}
&(1) \qquad xW_I \lessdot xW_{I\cup\{s\}} \qquad \text{if } s \notin I \text{ and } x \in W^{I\cup\{s\}},\\
&(2) \qquad xW_I \lessdot xw_{\circ,I}w_{\circ,I\smallsetminus\{s\}}W_{I\smallsetminus\{s\}} \qquad \text{if } s \in I,
\end{aligned}
$$

where $I \subseteq S$ and $x \in W^I$.

A Coxeter example

(1) $xW_I \lessdot xW_{I\cup\{s\}}$ if $s \notin I$ and $x \in W^{I\cup\{s\}}$
(2) $xW_I \lessdot xw_{\circ,I}w_{\circ,I\smallsetminus\{s\}}W_{I\smallsetminus\{s\}}$ if $s \in I$

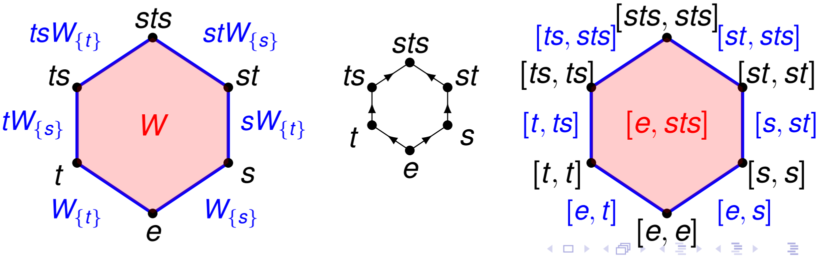


Facial intervals for Coxeter groups

Proposition (Björner, Las Vergas, Sturmfels, White, Ziegler ’93)

Let (W, S) be a finite Coxeter system and xW_I a standard parabolic coset. Then there exists a unique interval $[x, xw_{\circ,I}]$ in the weak order such that

$$xW_I = [x, xw_{\circ,I}].$$

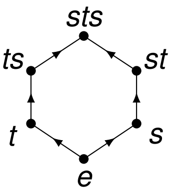


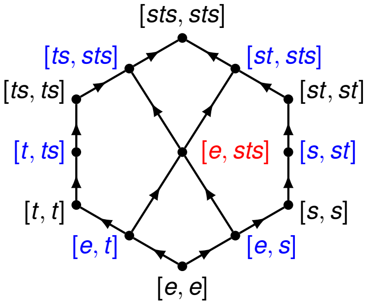
Facial weak order for Coxeter groups

Definition

Let $\leq_{F'}$ be the order on the Coxeter complex $\mathcal{P}_W$ defined by

$$xW_I \leq_{F'} yW_J \Leftrightarrow x \leq_R y \text{ and } xw_{\circ,I} \leq_R yw_{\circ,J}$$





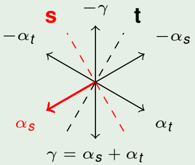
Visiting geometric lands

A system of roots

- Let $\mathcal{A}$ be a Coxeter arrangement.
- A *root system* is $\Phi := \{\pm\alpha_s \in V \mid H_s \in \mathcal{A}, ||\alpha_s|| = 1\}$
- We have $\Phi = \Phi^+ \sqcup \Phi^-$ decomposable into positive and negative roots.

Example

Let Γ_{A_2} : s —— t .

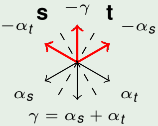


Inversions

Let (W, S) be a Coxeter system.
Define *(left) inversion sets* as the set $\mathbf{N}(w) := \Phi^+ \cap w(\Phi^-)$.

Example

Let Γ_{A_2} : 

, with Φ given by the roots 

$$\begin{aligned} \mathbf{N}(ts) &= \Phi^+ \cap ts(\Phi^-) \\ &= \Phi^+ \cap \{\alpha_t, \gamma, -\alpha_s\} \\ &= \{\alpha_t, \gamma\} \end{aligned}$$

Inversions

Let (W, S) be a Coxeter system.
Define *(left) inversion sets* as the set $\mathbf{N}(w) := \Phi^+ \cap w(\Phi^-)$.

Example

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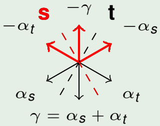

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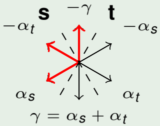

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Example

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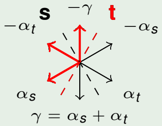
, with Φ given by the roots 

$$
\begin{aligned}
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&= \Phi^+ \cap \{\alpha_t, \gamma, -\alpha_s\} \\
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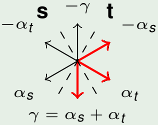
Inversions

Let (W, S) be a Coxeter system.
Define *(left) inversion sets* as the set $\mathbf{N}(w) := \Phi^+ \cap w(\Phi^-)$.

Example

Let Γ_{A_2} :  , with Φ given by the roots



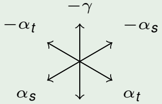
$$\begin{aligned}\mathbf{N}(ts) &= \Phi^+ \cap ts(\Phi^-)\\ &= \Phi^+ \cap \{\alpha_t, \gamma, -\alpha_s\}\\ &= \{\alpha_t, \gamma\}\end{aligned}$$

Weak order = Inversion sets

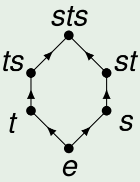
Given $w, u \in W$ then $w \leq_R u$ if and only if $\mathbf{N}(w) \subseteq \mathbf{N}(u)$.

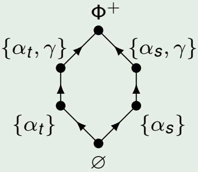
Example

Let Γ_{A_2} : 

, with Φ given by the roots 

$\gamma = \alpha_s + \alpha_t$





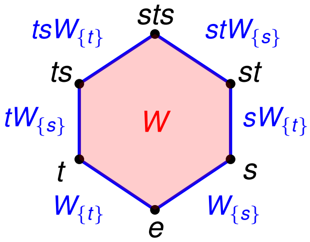
Root inversions

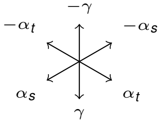
Definition (Root Inversion Set)

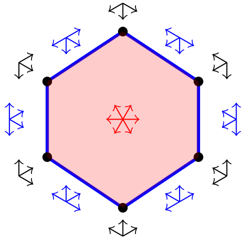
Let xW_I be a standard parabolic coset. The *root inversion set* is the set

$$\mathbf{R}(xW_I) := x(\Phi^- \cup \Phi_I^+)$$

Note that $N(x) = \mathbf{R}(xW_\varnothing) \cap \Phi^+$.



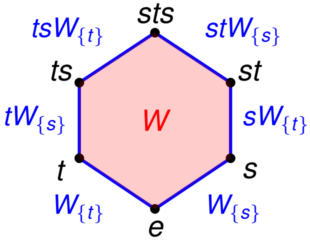


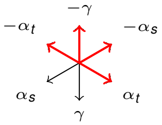


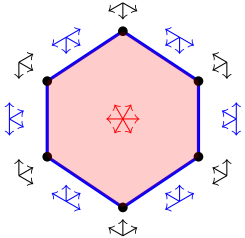
Root inversions

Example

$$\begin{aligned}\mathbf{R}(sW_{\{t\}}) &= s(\Phi^- \cup \Phi^+_{\{t\}}) \\ &= s(\{-\alpha_s, -\alpha_t, -\gamma\} \cup \{\alpha_t\}) \\ &= \{\alpha_s, -\gamma, -\alpha_t, \gamma\}\end{aligned}$$







Root inversions

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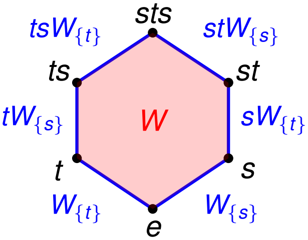

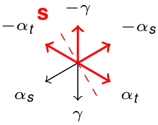

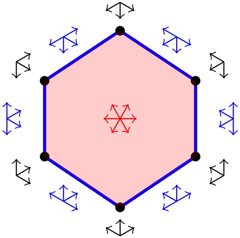

Root inversions

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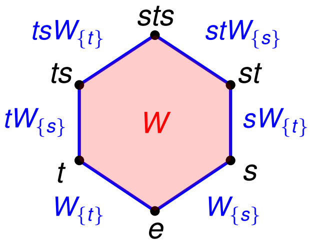

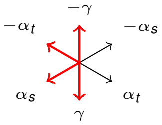

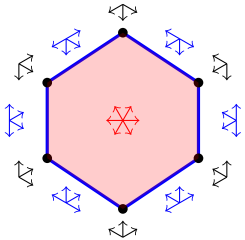

Root inversions

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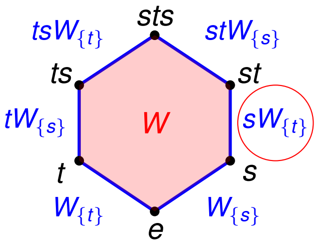
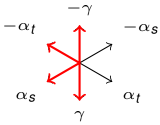
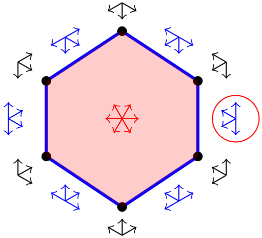




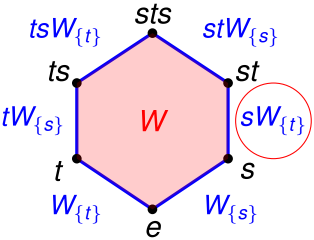


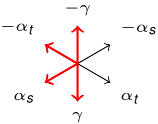
Root inversions

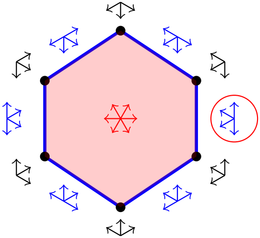
Proposition (D., Hohlweg, Pilaud '18)

Let xW_I be a standard parabolic coset of W. Then

$$\textbf{inner primal cone}\left(\mathbf{F}(xW_I)\right) = \text{cone}\left(\mathbf{R}(xW_I)\right).$$







Equivalent definitions

Theorem (D., Hohlweg, Pilaud '18)

Let (W, S) be a finite Coxeter system. The following conditions are equivalent for two standard parabolic cosets xW_I and yW_J in the Coxeter complex $\mathcal{P}_W$

1. $xW_I \leq_F yW_J$
2. $\mathbf{R}(xW_I) \smallsetminus \mathbf{R}(yW_J) \subseteq \Phi^-$ *and* $\mathbf{R}(yW_J) \smallsetminus \mathbf{R}(xW_I) \subseteq \Phi^+$.
3. $x \leq_R y$ *and* $xw_{\circ,I} \leq_R yw_{\circ,J}$.

Facial weak order lattice

Theorem (D., Hohlweg, Pilaud '18)

The facial weak order $(\mathcal{P}_W, \leq_F)$ is a lattice with the meet and join of two standard parabolic cosets xW_I and yW_J given by:

$$xW_I \wedge yW_J = z_\wedge W_{K_\wedge},$$
$$xW_I \vee yW_J = z_\vee W_{K_\vee}.$$

where,

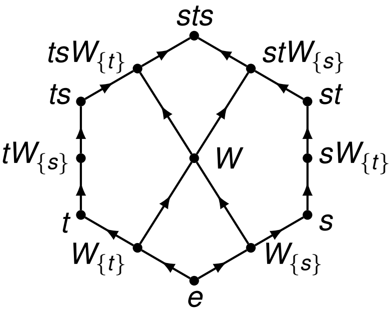
$z_\wedge = x \wedge y$ *and* $K_\wedge = D_L(z_\wedge^{-1}(xw_{\circ,I} \wedge yw_{\circ,J}))$, *and*

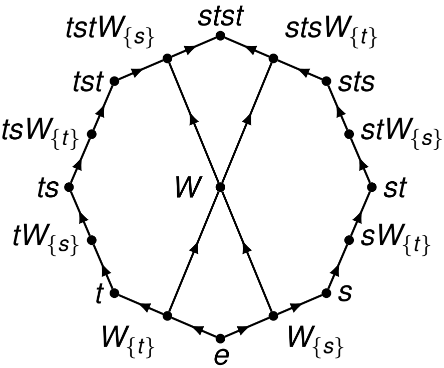
$z_\vee = xw_{\circ,I} \vee yw_{\circ,J}$ *and* $K_\vee = D_L(z_\vee^{-1}(x \vee y))$

Corollary (D., Hohlweg, Pilaud '18)

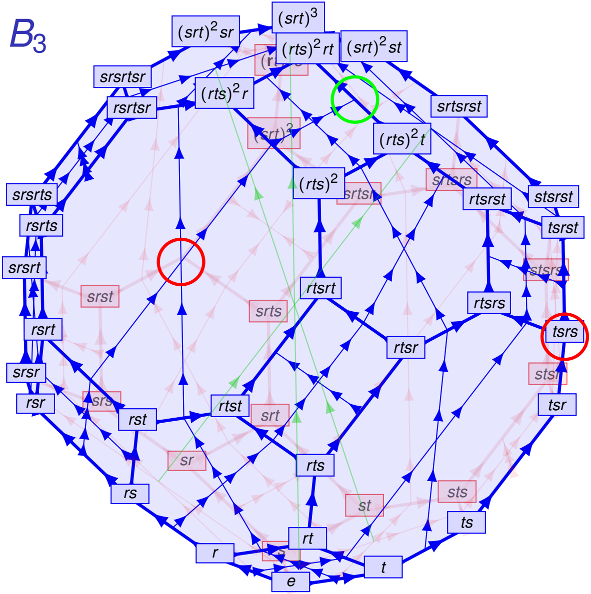
The weak order is a sublattice of the facial weak order lattice.

Example: A_2 and B_2





Example: B_3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Back to arrangements

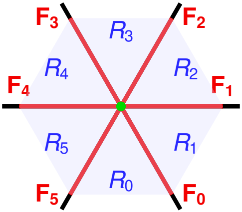
One step at a time

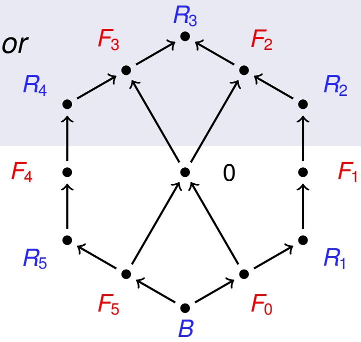
Proposition (D., Hohlweg, McConville, Pilaud, '19+)

For $F, G \in \mathscr{F}_{\mathcal{A}}$ if

1. $|\dim(F) - \dim(G)| = 1$
2. - $F \subseteq G$ *and* $M_F = M_G$, *or*
   - $G \subseteq F$ *and* $m_F = m_G$.

then $F \lessdot G$.





Zonotopes

- *Zonotope $Z_{\mathcal{A}}$* is the convex polytope:

$$Z_{\mathcal{A}} := \left\{ v \in V \mid \; v = \sum_{i=1}^{k} \lambda_i e_i, \text{ such that } |\lambda_i| \leq 1 \text{ for all } i \right\}$$

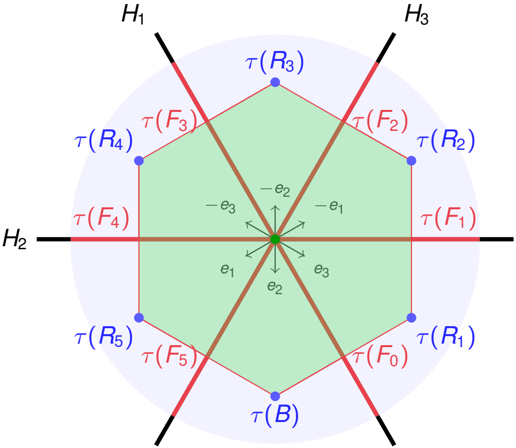
Theorem (Edelman '84, McMullen '71)

There is a bijection between $\mathscr{F}_{\mathcal{A}}$ and the nonempty faces of $Z_{\mathcal{A}}$ given by the map

$$\tau(F) = \left\{ v \in V \mid \; v = \sum_{F(H_i)=0} \lambda_i e_i + \sum_{F(H_j)\neq 0} \mu_j e_j \right\}$$

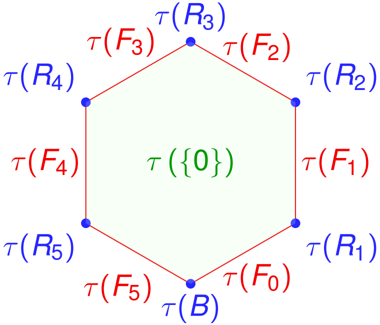
$$\textit{where } |\lambda_i| \leq 1 \textit{ for all } i \textit{ and } \mu_j = F(H_j)$$

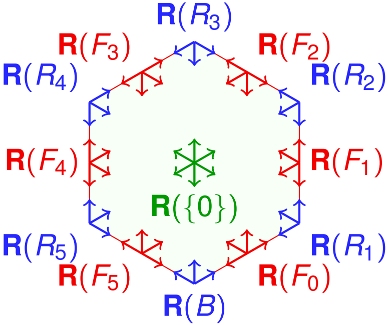
Zonotope example



Root inversions for arrangements

- *roots* $\Phi_{\mathcal{A}} := \{\pm e_1, \pm e_2, \dots, \pm e_k\}$
- *root inversion set*
  $\mathbf{R}(F) := \{e \in \Phi_{\mathcal{A}} \mid \langle x, e \rangle \leq 0 \text{ for some } x \in \text{int}(F)\}$.



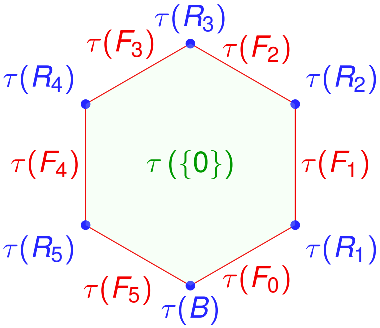


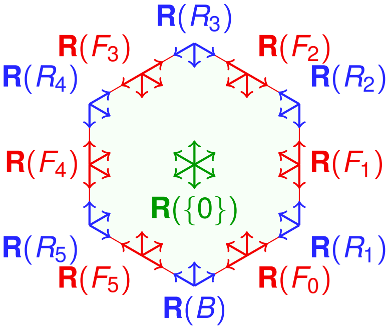
Root inversions for arrangements

Proposition (D., Hohlweg, McConville, Pilaud '19+)

Let F be a face. Then

$$\textbf{inner primal cone}\,(\tau(F)) = \text{cone}\,(\mathbf{R}(F))\,.$$



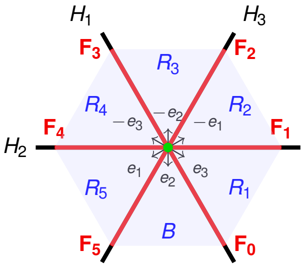


Covectors

- *covector* - a vector in $\{-, 0, +\}^{\mathcal{A}}$ with signs relative to hyperplanes.
- $\mathcal{L} \subseteq \{-, 0, +\}^{\mathcal{A}}$ - set of covectors

Example

$$F_4(H_1) = +; \ F_4(H_2) = 0; \ F_4(H_3) = - \qquad F_4 \leftrightarrow (+, 0, -)$$



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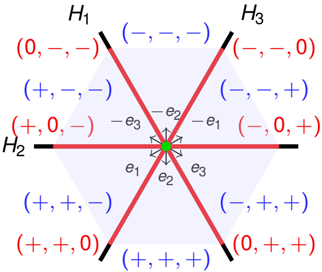
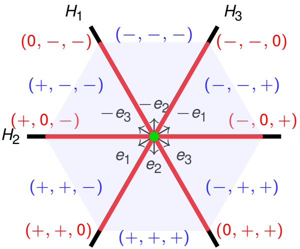


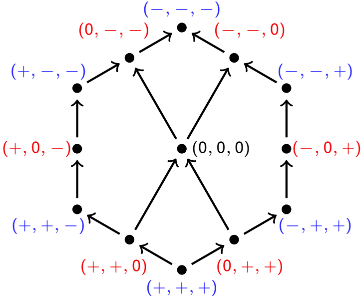
Covector Definition

Definition

For $X, Y \in \mathcal{L}$:

$$X \leq_{\mathcal{L}} Y \Leftrightarrow X(H) \geq Y(H) \quad \forall H \text{ with } - < 0 < +$$





Equivalent definitions

Theorem (D., Hohlweg, McConville, Pilaud '19+)

Let $\mathcal{A}$ be a hyperplane arrangement. For $F, G \in \mathscr{F}_{\mathcal{A}}$ the following are equivalent:

- *$m_F \leq_{\mathrm{PR}} m_G$ and $M_F \leq_{\mathrm{PR}} M_G$ in poset of regions $\mathrm{PR}(\mathcal{A}, B)$.*
- *There exists a chain of covers in $\mathrm{FW}(\mathcal{A}, B)$ such that*

$$F = F_1 \lessdot F_2 \lessdot \cdots \lessdot F_n = G$$

- *$F \leq_{\mathcal{L}} G$ in terms of covectors ($F(H) \geq G(H)\ \forall H \in \mathcal{A}$)*
- *$\mathbf{R}(F) \setminus \mathbf{R}(G) \subseteq \Phi^-_{\mathcal{A}}$ and $\mathbf{R}(G) \setminus \mathbf{R}(F) \subseteq \Phi^+_{\mathcal{A}}$.*

Facial weak order lattice

Theorem (D., Hohlweg, McConville, Pilaud '19+)

The facial weak order $\mathrm{FW}(\mathcal{A}, B)$ *is a lattice when* $\mathrm{PR}(\mathcal{A}, B)$ *is a lattice.*

Corollary (D., Hohlweg, McConville, Pilaud '19+)

The lattice of regions is a sublattice of the facial weak order lattice when $\mathcal{A}$ *is simplicial.*

Properties of the FWO

Semi-distributive duality

- The *dual* of a poset P is the poset P^{op} where $x \leq y$ in P iff $y \leq x$ in P^{op}. A poset is *self-dual* if $P \cong P^{op}$.
- A lattice is *semi-distributive* if $x \vee y = x \vee z$ implies $x \vee y = x \vee (y \wedge z)$ and similarly for the meets.

Theorem (D., Hohlweg, McConville, Pilaud '19+)

The facial weak order $\mathrm{FW}(\mathcal{A}, B)$ *is self-dual. If furthermore,* $\mathcal{A}$ *is simplicial,* $\mathrm{FW}(\mathcal{A}, B)$ *is a semi-distributive lattice.*

Join-irreducible elements

- An element is *join-irreducible* if and only if it covers exactly one element.

Proposition (D., Hohlweg, McConville, Pilaud '19+)

If $\mathcal{A}$ is a simplicial arrangement and F a face with facial interval $[m_F, M_F]$. Then F is join-irreducible in $\mathrm{FW}(\mathcal{A}, B)$ if and only if M_F is join-irreducible in $\mathrm{PR}(\mathcal{A}, B)$ and $\mathrm{codim}(F) \in \{0, 1\}$

Proposition (D., Hohlweg, Pilaud '18)

Let (W, S) be a finite Coxeter system. A standard parabolic coxet xW_I is join-irreducible in the facial weak order if and only if we have one of the two following cases

- $I = \varnothing$ *and x is join-irreducible in the right weak order, or*
- $I = \{s\}$ *and xs is join-irreducible in the right weak order.*

Möbius function

Recall that the *Möbius function* of a poset $(P, \leq)$ is the function $\mu : P \times P \to \mathbb{Z}$ defined inductively by

$$\mu(x, y) := \begin{cases} 1 & \text{if } x = y, \\ -\sum_{x \leq z < y} \mu(x, z) & \text{if } x < y, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition (D., Hohlweg, Pilaud '18)

The Möbius function of the facial weak order of a finite Coxeter system (W, S) is given by

$$\mu(eW_\varnothing, yW_J) = \begin{cases} (-1)^{|J|}, & \text{if } y = e, \\ 0, & \text{otherwise.} \end{cases}$$

Möbius function

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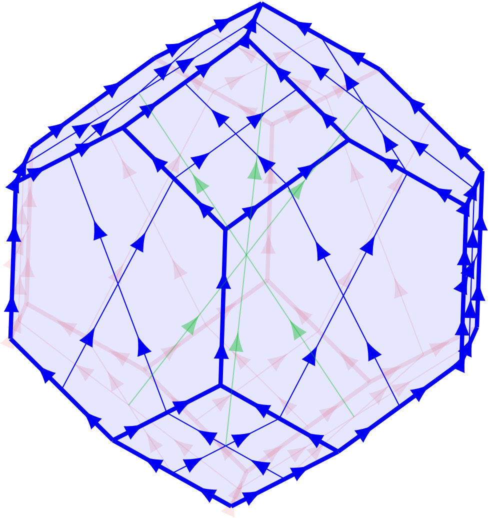
Proposition (D., Hohlweg, McConville, Pilaud '19+)

Let X and Y be faces of $\mathcal{A}$ such that $X \leq Y$ and let $Z = X \cap Y$.

$$\mu(X,Y) = \begin{cases} (-1)^{\mathrm{rk}(X)+\mathrm{rk}(Y)} & \text{if } X \leq Z \leq Y \text{ and } Z = X_{-Z} \cap Y \\ 0 & \text{otherwise} \end{cases}$$

Further Works

- Can we explicitly state the join/meet of two elements for hyperplane arrangements?
- When is the facial weak order congruence uniform?
- How many maximal chains are there?
- What is the order dimension?
- Can we generalize this to polytopes?



Thank you!