The facial weak order in hyperplane arrangements

Aram Dermenjian
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Joint with: Christophe Hohlweg, Thomas McConville and Vincent Pilaud
(UQAM) (MSRI) (LIX)

8 August 2019

On this day in 1902 Paul Dirac was born.
And in 1900 Hilbert revealed 10 of his 23 problems at 2nd ICM in Paris.

“We hear within us the perpetual call: There is a problem. Seek its solution. You can find it by pure reason, for in mathematics there is no ’ignorabimus’. ” - Hilbert

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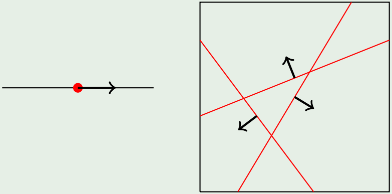
History and Background - Hyperplanes

- $(V, \langle\cdot,\cdot\rangle)$ - n-dim real Euclidean vector space.
- A *hyperplane H* is $\operatorname{codim}\ 1$ subspace of V with normal e_H.

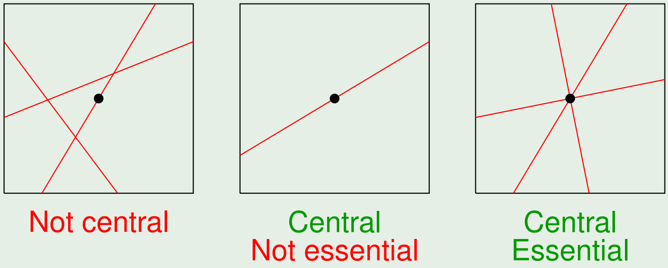
Example



History and Background - Arrangements

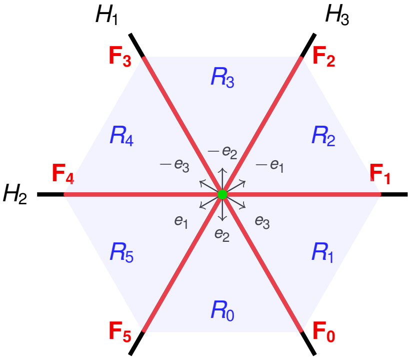
- A *hyperplane arrangement* is $\mathcal{A} = \{H_1, H_2, \ldots, H_k\}$.
- $\mathcal{A}$ is *central* if $\{0\} \subseteq \bigcap \mathcal{A}$.
- Central $\mathcal{A}$ is *essential* if $\{0\} = \bigcap \mathcal{A}$.

Example



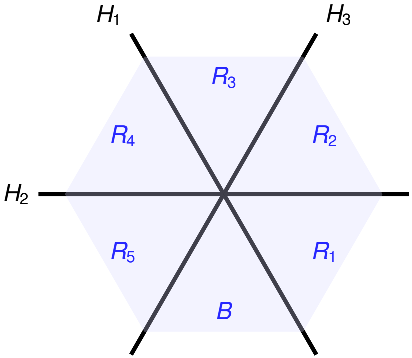
History and Background - Arrangements

- *Regions* $\mathscr{R}_{\mathcal{A}}$ - connected components of V without $\mathcal{A}$.
- *Faces* $\mathscr{F}_{\mathcal{A}}$ - intersections of closures of some regions.



History and Background - Poset of regions

- *Base region $B \in \mathscr{R}_{\mathcal{A}}$* - some fixed region
- *Separation set for $R \in \mathscr{R}_{\mathcal{A}}$*
  $S(R) := \{H \in \mathcal{A} \mid \; H \text{ separates } R \text{ from } B\}$



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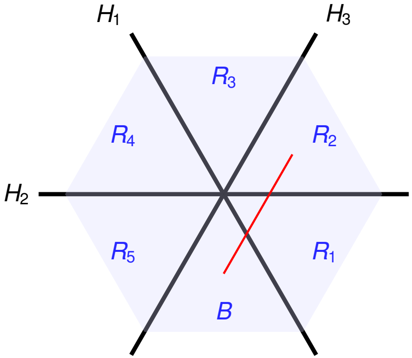

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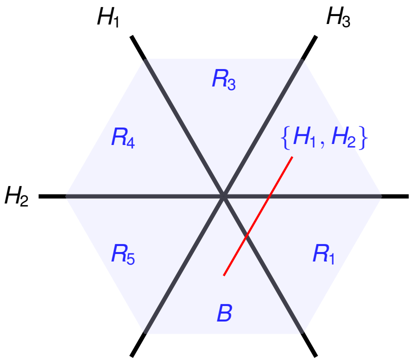

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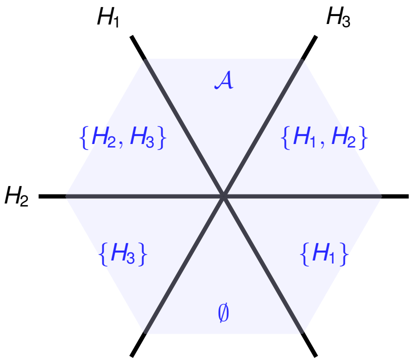
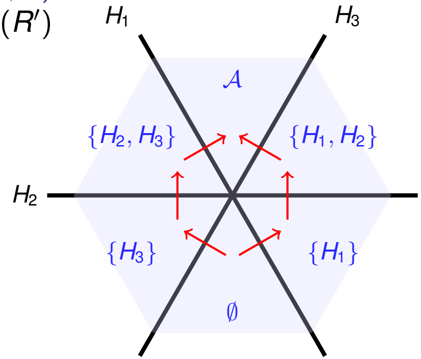


History and Background - Poset of regions

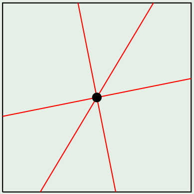
- *Base region* $B \in \mathscr{R}_{\mathcal{A}}$ - some fixed region
- *Separation set for* $R \in \mathscr{R}_{\mathcal{A}}$
  $S(R) := \{H \in \mathcal{A} \mid H \text{ separates } R \text{ from } B\}$
- *Poset of Regions* $\mathrm{PR}(\mathcal{A}, B)$ where
  $R \leq_{\mathrm{PR}} R' \Leftrightarrow S(R) \subseteq S(R')$



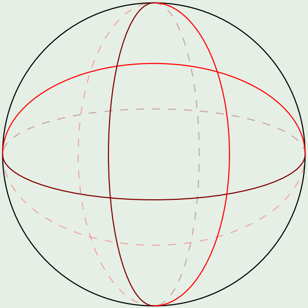
History and Background - Poset of regions

- A region R is *simplicial* if normal vectors for boundary hyperplanes are linearly independent.
- $\mathcal{A}$ is *simplicial* if all $\mathscr{R}_{\mathcal{A}}$ simplicial.

Example



Simplicial



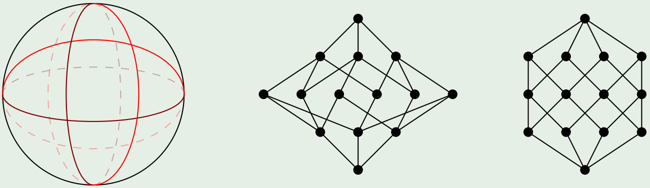
Not simplicial

History and Background - Poset of regions

Theorem (Björner, Edelman, Ziegler '90)

If $\mathcal{A}$ is simplicial then $\mathrm{PR}(\mathcal{A}, B)$ is a lattice for any $B \in \mathscr{R}_{\mathcal{A}}$. If $\mathrm{PR}(\mathcal{A}, B)$ is a lattice then B is simplicial.

Example



History and Background - Poset of regions

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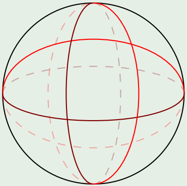
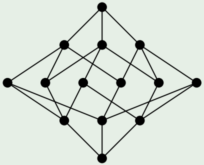
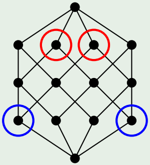






Motivation

- **2001:** Krob, Latapy, Novelli, Phan, and Schwer extended the weak order of type *A* Coxeter groups to all the faces of its associated arrangement.
- **2006:** Palacios and Ronco extended this new order to Coxeter groups of all types using cover relations.
- **2016:** D, Hohlweg and Pilaud gave a global equivalent to this extension and showed it's a lattice.

Motivation

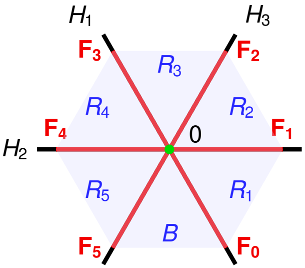
- **2001:** Krob, Latapy, Novelli, Phan, and Schwer extended the weak order of type A Coxeter groups to all the faces of its associated arrangement.
- **2006:** Palacios and Ronco extended this new order to Coxeter groups of all types using cover relations.
- **2016:** D, Hohlweg and Pilaud gave a global equivalent to this extension and showed it's a lattice.
- Questions: Can we extend this to hyperplane arrangements? Can we find both local and global definitions? When do we actually get a lattice?

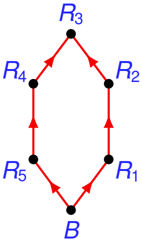
Facial Intervals

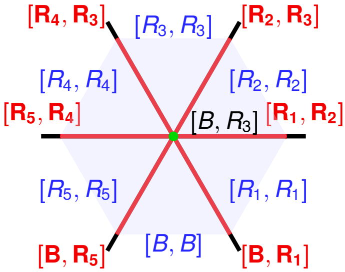
Proposition (Björner, Las Vergas, Sturmfels, White, Ziegler '93)

Let $\mathcal{A}$ be central with base region B. For every $F \in \mathscr{F}_{\mathcal{A}}$ there is a unique interval $[m_F, M_F]$ in $\mathrm{PR}(\mathcal{A}, B)$ such that

$$[m_F, M_F] = \left\{ R \in \mathscr{R}_{\mathcal{A}} \mid F \subseteq \overline{R} \right\}$$







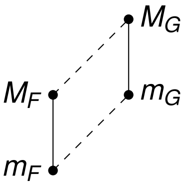
Facial Weak Order

Let $\mathcal{A}$ be a central hyperplane arrangement and B a base region in $\mathscr{R}_{\mathcal{A}}$.

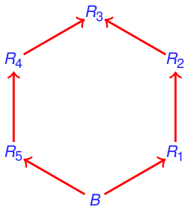
Definition

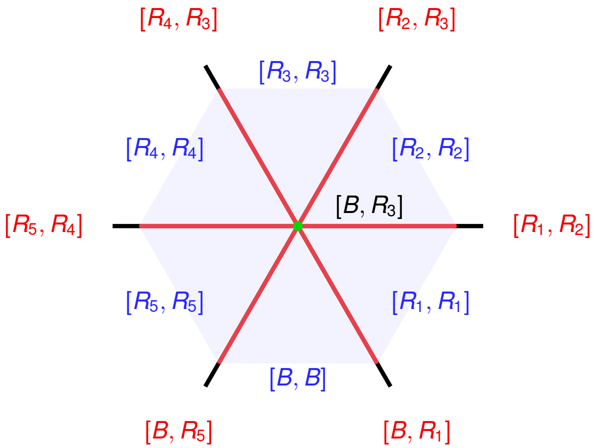
The *facial weak order* is the order $\mathrm{FW}(\mathcal{A}, B)$ on $\mathscr{F}_{\mathcal{A}}$ where for $F, G \in \mathscr{F}_{\mathcal{A}}$:

$$F \leq G \Leftrightarrow m_F \leq_{\mathrm{PR}} m_G \text{ and } M_F \leq_{\mathrm{PR}} M_G$$



Facial Weak Order - Example





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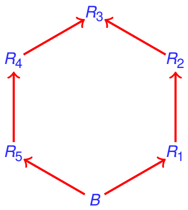

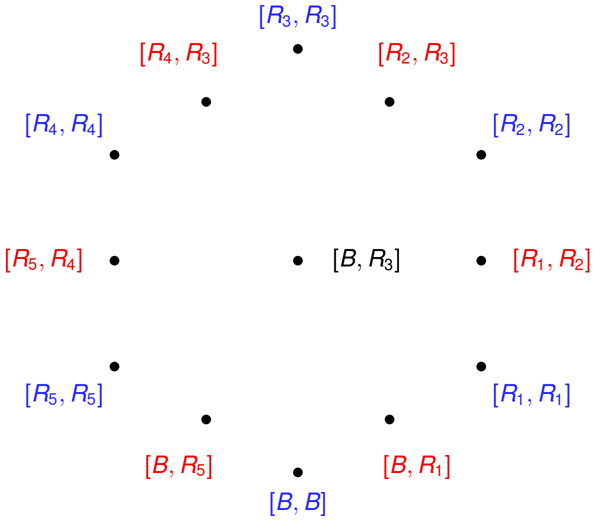

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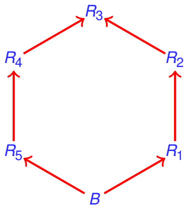

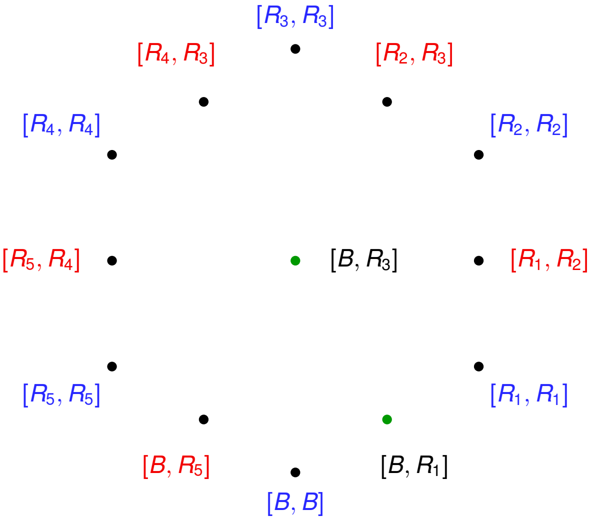

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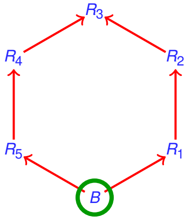

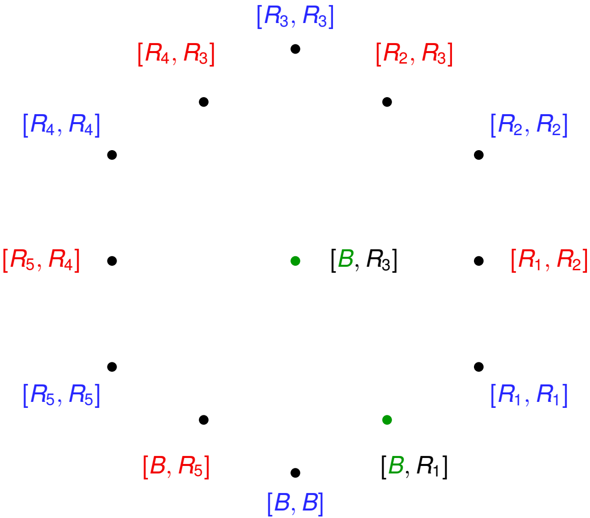

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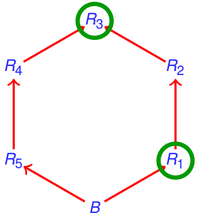

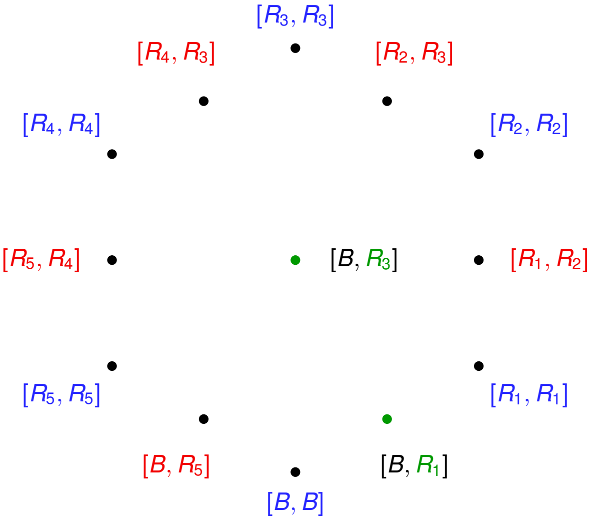

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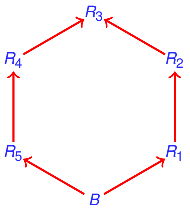
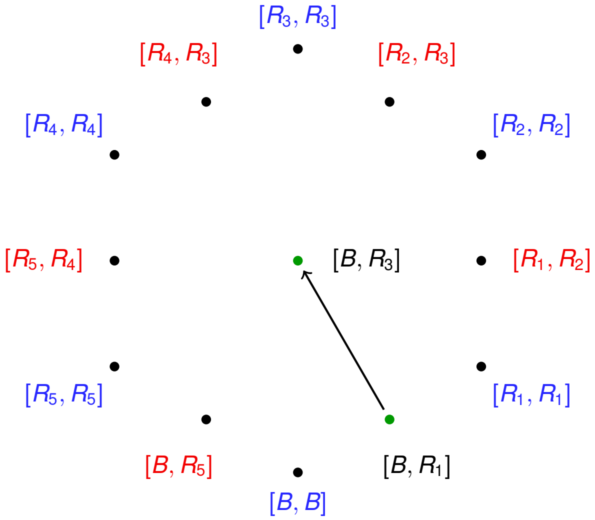
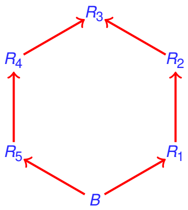
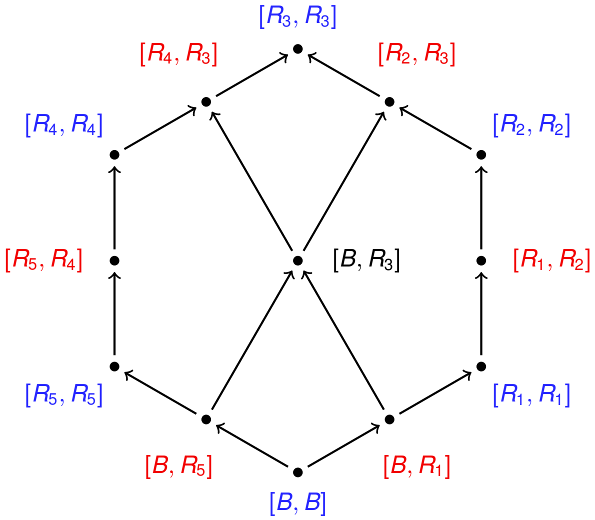




Facial Weak Order - Example





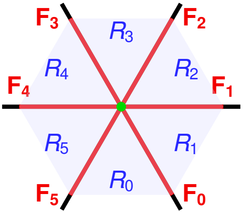
Cover Relations

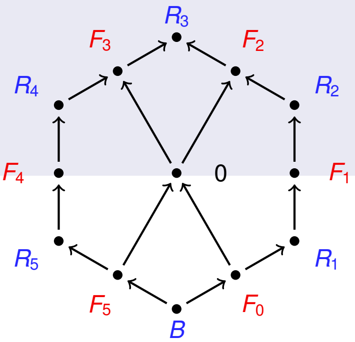
Proposition (D., Hohlweg, McConville, Pilaud, ’19+)

For $F, G \in \mathscr{F}_{\mathcal{A}}$ if

1. $F \le G$ *in* $\mathrm{FW}(\mathcal{A}, B)$
2. $|\dim(F) - \dim(G)| = 1$
3. $F \subseteq G$ *or* $G \subseteq F$

then $F \lessdot G$.



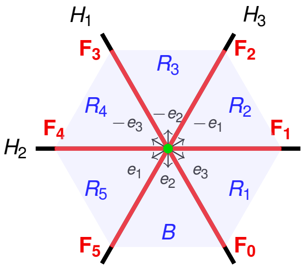


Covectors

- *covector* - a vector in $\{-,0,+\}^{\mathcal{A}}$ with signs relative to hyperplanes.
- $\mathcal{L} \subseteq \{-,0,+\}^{\mathcal{A}}$ - set of covectors

Example

$$F_4 \leftrightarrow (+,0,-) \qquad F_4(H_1) = +;\ F_4(H_2) = 0;\ F_4(H_3) = -$$



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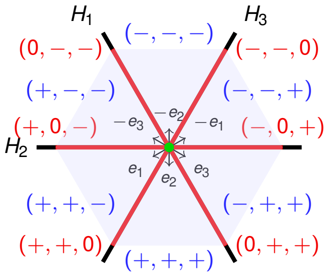
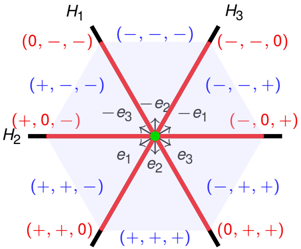


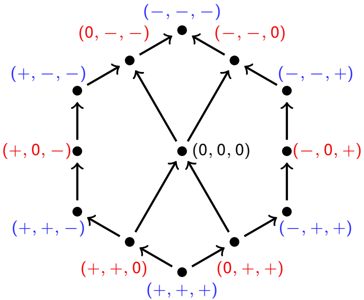
Covector Definition

Definition

For $X, Y \in \mathcal{L}$:

$$X \leq_{\mathcal{L}} Y \Leftrightarrow X(H) \geq Y(H) \quad \forall H \text{ with } - < 0 < +$$





Zonotopes

- *Zonotope $Z_{\mathcal{A}}$* is the convex polytope:

$$Z_{\mathcal{A}} := \left\{ v \in V \mid \; v = \sum_{i=1}^{k} \lambda_i e_i, \text{ such that } |\lambda_i| \leq 1 \text{ for all } i \right\}$$

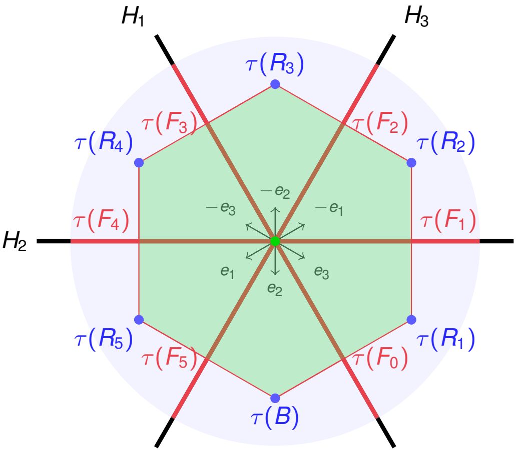
Theorem (Edelman '84, McMullen '71)

There is a bijection between $\mathscr{F}_{\mathcal{A}}$ and the nonempty faces of $Z_{\mathcal{A}}$ given by the map

$$\tau(F) = \left\{ v \in V \mid \; v = \sum_{F(H_i)=0} \lambda_i e_i + \sum_{F(H_j)\neq 0} \mu_j e_j \right\}$$

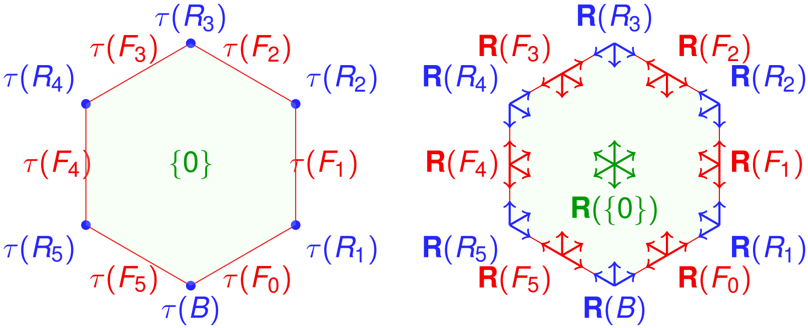
$$\textit{where } |\lambda_i| \leq 1 \textit{ for all } i \textit{ and } \mu_j = F(H_j)$$

Zonotope - Construction example



Root inversion sets

- *roots* $\Phi_{\mathcal{A}} := \{\pm e_1, \pm e_2, \ldots, \pm e_k\}$
- *root inversion set*
  $\mathbf{R}(F) := \{e \in \Phi_{\mathcal{A}} \mid \langle x, e \rangle \leq 0 \text{ for some } x \in F\}.$



Equivalent definitions

Theorem (D., Hohlweg, McConville, Pilaud '19+)

For $F, G \in \mathscr{F}_{\mathcal{A}}$ the following are equivalent:

- $m_F \leq_{\mathrm{PR}} m_G$ *and* $M_F \leq_{\mathrm{PR}} M_G$ *in poset of regions* $\mathrm{PR}(\mathcal{A}, B)$.
- *There exists a chain of covers in* $\mathrm{FW}(\mathcal{A}, B)$ *such that*

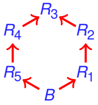
$$F = F_1 \lessdot F_2 \lessdot \cdots \lessdot F_n = G$$

- $F \leq_{\mathcal{L}} G$ *in terms of covectors* $(F(H) \geq G(H)\ \forall H \in \mathcal{A})$
- $\mathbf{R}(F) \backslash \mathbf{R}(G) \subseteq \Phi^{-}_{\mathcal{A}}$ *and* $\mathbf{R}(G) \backslash \mathbf{R}(F) \subseteq \Phi^{+}_{\mathcal{A}}$.

Warning!

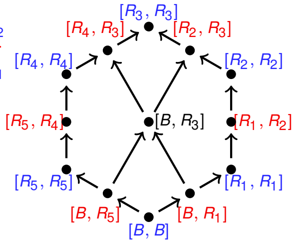
Next slide contains a lot of data. . . please procede with caution.

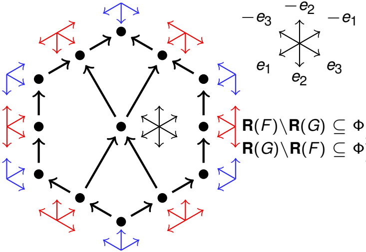
Equivalence for type A_2 Coxeter arrangement



$m_F \leq_{PR} m_G$

$M_F \leq_{PR} M_G$





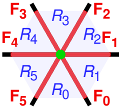
$\mathbf{R}(F) \setminus \mathbf{R}(G) \subseteq \Phi^-_{\mathcal{A}}$

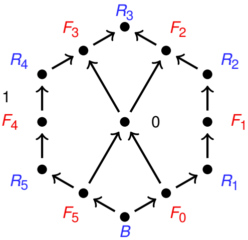
$\mathbf{R}(G) \setminus \mathbf{R}(F) \subseteq \Phi^+_{\mathcal{A}}$

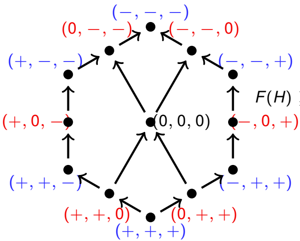
$F \leq G$

$F \subseteq G$ or $G \subseteq F$

$|\dim F - \dim G| = 1$







$F(H) \geq G(H)\ \forall H \in \mathcal{A}$

Facial weak order lattice

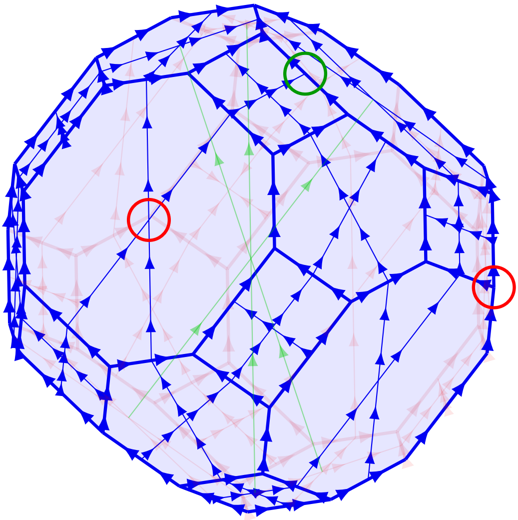
Theorem (D., Hohlweg, McConville, Pilaud '19+)

The facial weak order $\mathrm{FW}(\mathcal{A}, B)$ *is a lattice when* $\mathcal{A}$ *is simplicial.*

Corollary (D., Hohlweg, McConville, Pilaud '19+)

The lattice of regions is a sublattice of the facial weak order lattice when $\mathcal{A}$ *is simplicial.*

Example: B_3 Coxeter arrangement



Properties of the facial weak order

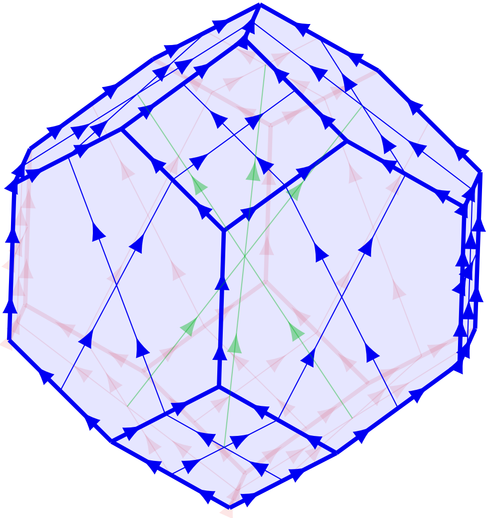
Theorem (D., Hohlweg, McConville, Pilaud '19+)

- $\mathrm{FW}(\mathcal{A}, B)$ *is self-dual.*
- $\mathcal{A}$ *simplicial implies* $\mathrm{FW}(\mathcal{A}, B)$ *is semi-distributive.*
- $\mathcal{A}$ *simplicial and* $X \in \mathscr{F}_{\mathcal{A}}$ *then* X *is join-irreducible in* $\mathrm{FW}(\mathcal{A}, B)$ *if and only if* M_X *is join-irreducible in* $\mathrm{PR}(\mathcal{A}, B)$ *and* $\mathrm{codim}(X) \in \{0, 1\}$
- *Möbius function:* $X, Y \in \mathscr{F}_{\mathcal{A}}$ *let* $Z = X \cap Y$.

$$\mu(X, Y) = \begin{cases} (-1)^{\mathrm{rk}(X)+\mathrm{rk}(Y)} & \text{if } X \leq Z \leq Y \text{ and } Z = X_{-Z} \cap Y \\ 0 & \text{otherwise} \end{cases}$$

Further Works

- Can we explicitly state the join/meet of two elements?
- When is the facial weak order congruence uniform?
- How many maximal chains are there?
- What is the order dimension?
- Can we generalize this to polytopes?



Thank you!