Sign Variation and Descents

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Joint with: Nantel Bergeron and John Machacek

York University

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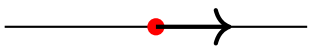
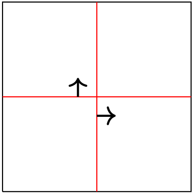
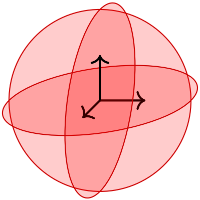
Sign vectors

- A *sign vector* is a vector in $\mathcal{V}_n = \{+, 0, -\}^n$.
- Think of these as vectors associated to faces in $\mathbb{R}^n$.

Example

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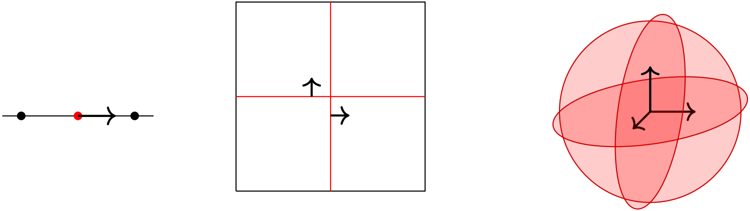

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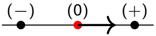

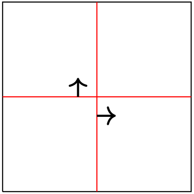

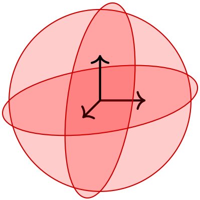

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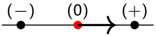

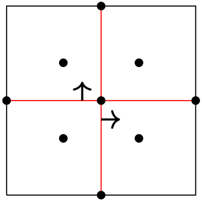

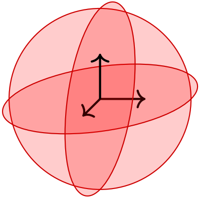

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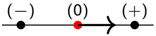

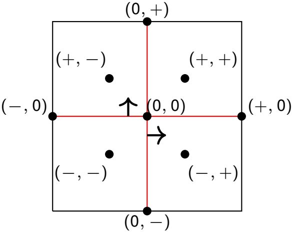

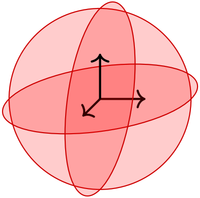

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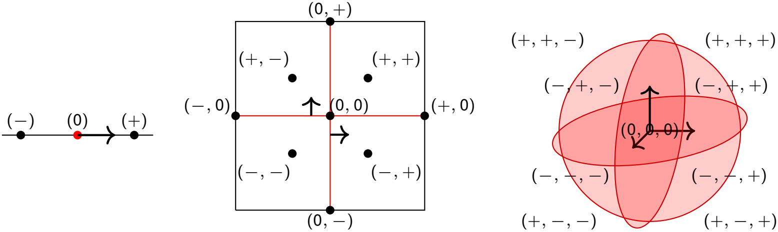
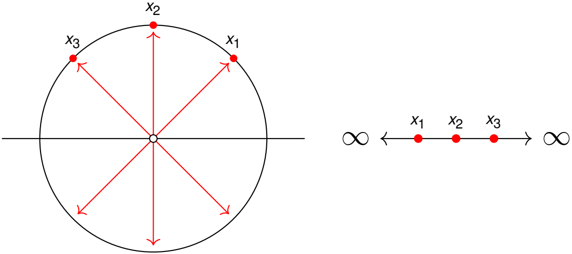


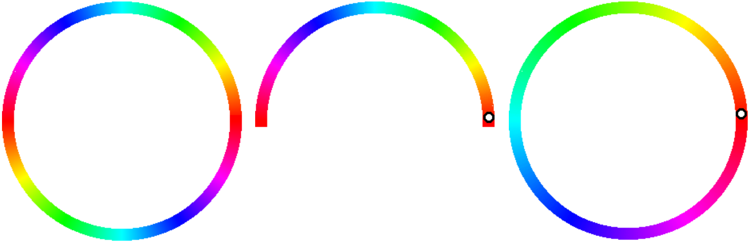
Real Projective Space

- *Real Projective space* $\mathbb{RP}^n$ is quotient of $\mathbb{R}^{n+1} \setminus \{0\}$ under equivalence relation $x \sim \lambda x$ for $\lambda \in \mathbb{R}$.

$\mathbb{RP}^1$



Real Projective Space - $\mathbb{RP}^1$



Images from: math.stackexchange.com by Zev Chonoles

Projective sign vectors

- Let $\mathcal{PV}_n$ be the set of sign vectors in $\mathbb{RP}^{n-1}$.
- In other words, for $\omega \in \mathcal{V}_n$, then $\omega \sim \omega'$ iff $\omega = \omega'$ or $\omega = -\omega'$ and

$$\mathcal{PV}_n = \left(\mathcal{V}_n \backslash \{0\}^n\right) / \sim .$$

Example

$$\mathcal{V}_1 = \{(+), (0), (-)\}$$

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Ordering projective sign vectors

- Let P_n denote the poset $(\mathcal{PV}_n, <)$ where for $\omega, \omega' \in \mathcal{PV}_n$:

$$\omega' < \omega \iff \pm\omega' \subseteq \omega$$

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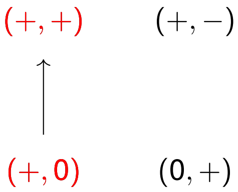

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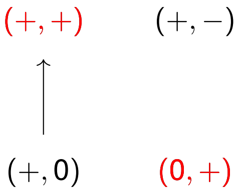

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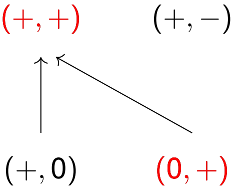

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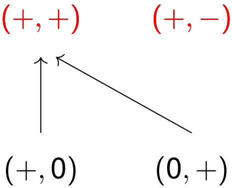

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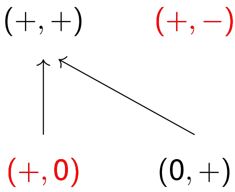

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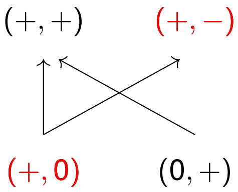

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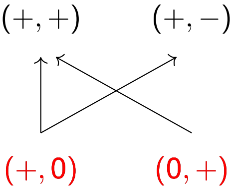

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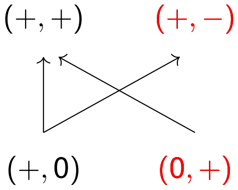

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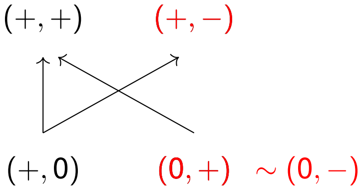

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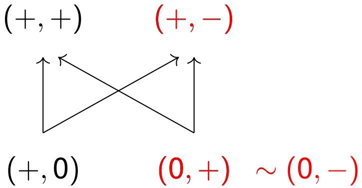

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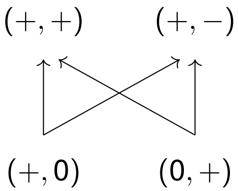

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- *Simplicial complex* Δ - A collection of sets s.t. $\sigma \in \Delta$ and $\tau \subseteq \sigma$ implies $\tau \in \Delta$.
- The sets are called *faces*. Maximal sets are called *facets*.
- *Order complex* $\Delta(P)$ of a poset P - Simplicial complex where faces are chains in P.

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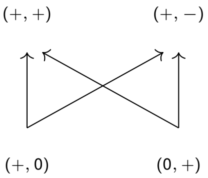

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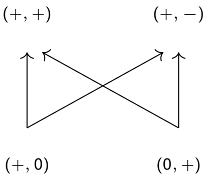

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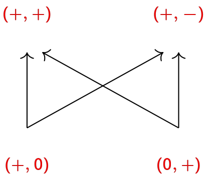

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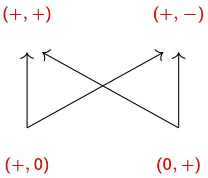

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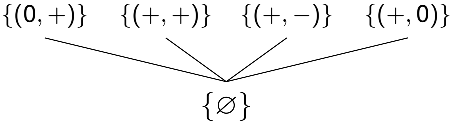
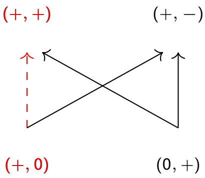


Order complex (of a poset)

- *Simplicial complex* Δ - A collection of sets s.t. $\sigma \in \Delta$ and $\tau \subseteq \sigma$ implies $\tau \in \Delta$.
- The sets are called *faces*. Maximal sets are called *facets*.
- *Order complex* $\Delta(P)$ of a poset P - Simplicial complex where faces are chains in P.

Example



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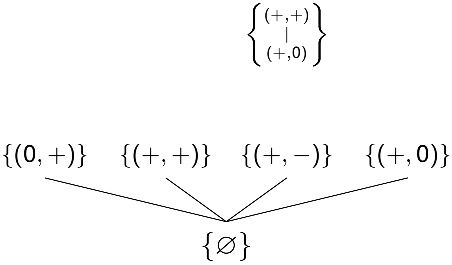

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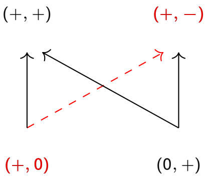

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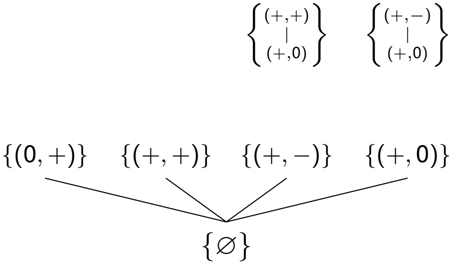

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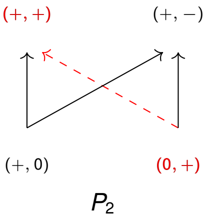

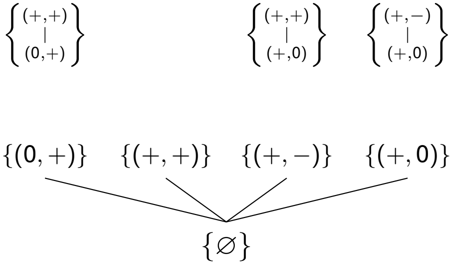

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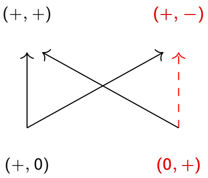

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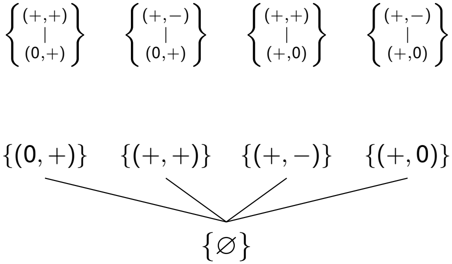

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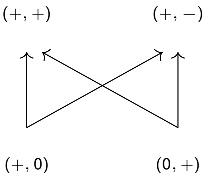

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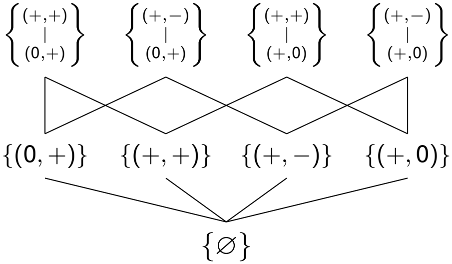
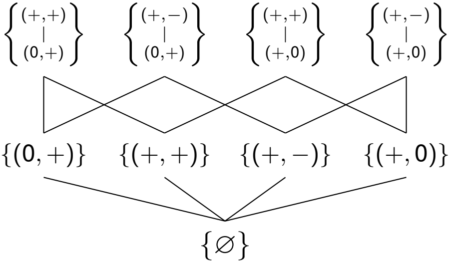


$\Delta(P_2)$

f-vector

- Δ a d-dim simplicial complex.
- f_i = number of i-dim faces
- *f-vector* is vector faces: $f(\Delta) = (f_{-1}, f_0, f_1, \ldots, f_d)$.
- $f(\Delta(P))$ is number of elements in each row.

Example



$$f(\Delta(P_2)) = (1, 4, 4)$$

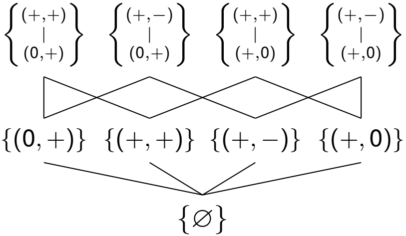
h-vectors

- Δ a d-dim simplicial complex with $f(\Delta) = (f_{-1}, f_0, \ldots, f_d)$.

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} f_{i-1}.$$

- *h-vector* is vector of h_ks: $h(\Delta) = (h_0, h_1, \ldots, h_{d+1})$.

Example



$$f(\Delta(P_2)) = (1, 4, 4)$$

$$h(\Delta(P_2)) = (1, 2, 1)$$

How can we find the *h*-vector?

Theorem (Stanley 1992(?))

If a simplicial complex Δ is Cohen-Macaulay, its h-vector has nonnegative entries.

Theorem (Machacek 2019)

The order complex $\Delta(P_n)$ is Cohen-Macaulay.

Questions

- *Is there a nice way to compute the h-vector of $\Delta(P_n)$?*

Partitionable simplicial complex

Conjecture (Stanley 1979, Garsia 1980; Counterexample Duval, Goeckner, Klivans, Martin 2016)

Every Cohen-Macaulay simplicial complex is partitionable.

Proposition (Stanley)

If Δ is partitionable, then the partitioning gives us the h-vector.

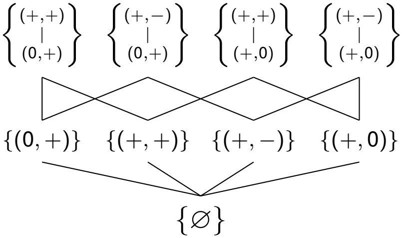
Partitionable

A simplicial complex Δ is *partitionable* if

$$\Delta = \bigsqcup [G_i, F_i]$$

where F_i is a facet.

Example



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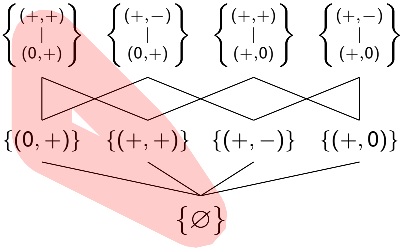

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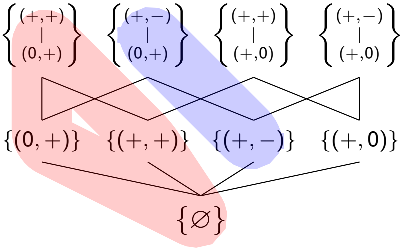

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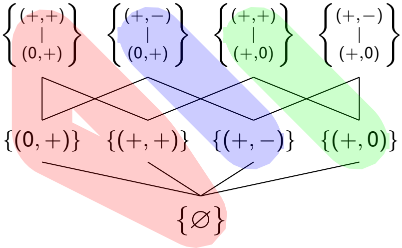

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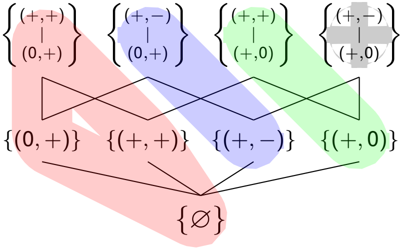


$\Delta(P_2)$

Partitionable

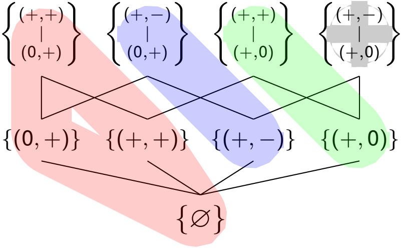
Proposition (Stanley)

Let Δ be a partitionable simplicial complex with partitioning $\Delta = \sqcup [G_i, F_i]$ where F_i is a facet. Then

$$h_i(\Delta) = |\{j \ : \ |G_j| = i\}|.$$

Example

$$h(\Delta(P_2)) = (h_0, h_1, h_2)$$



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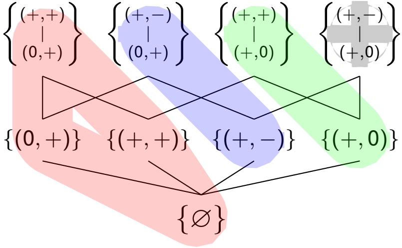

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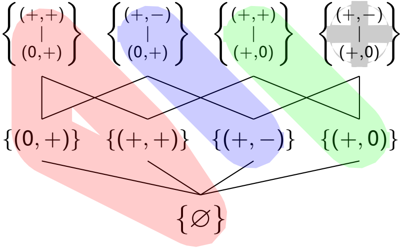

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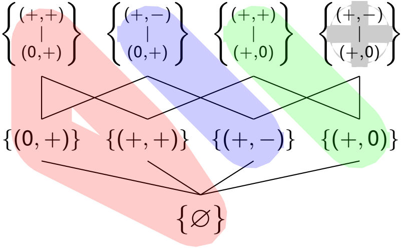

$\Delta(P_2)$

Main Theorem

Let D_n be a type D Coxeter group and let des_B denote the type B descent set of an element $\pi \in D_n$.

Theorem (Bergeron, D., Machacek 2020BP)

The order complex $\Delta(P_n)$ is partitionable. Moreover,

$$h_i(\Delta(P_n)) = |\{\pi \in D_n \ : \ |\mathrm{des}_B(\pi)| = i\}|$$

for each $0 \leq i \leq n$.

Questions:

At this point you should be asking yourself:

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Answer

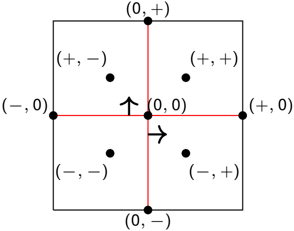
After the break!

Quick Recap

- Sign Vectors

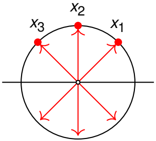
$$\mathcal{V}_n = \{+, 0, -\}^n$$

Example



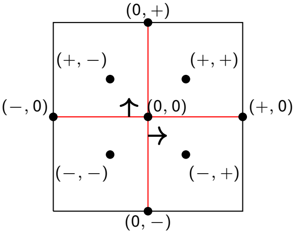
Quick Recap

- Projective Sign Vectors



$$\mathcal{PV}_n = \left(\mathcal{V}_n \setminus \{0\}^n\right) / \sim$$
$$\cong \{\omega \in \mathcal{V}_n \ : \ \text{First non-zero entry of } \omega \text{ is } +\}\,.$$

Example



$(+,+)$ $(+,-)$

$(+,0)$ $(0,+)$

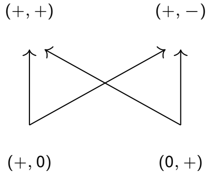
Quick Recap

- Order $P_n = (\mathcal{PV}_n, <)$

where for $\omega, \omega' \in \mathcal{PV}_n$:

$$\omega' < \omega \iff \pm\omega' \subseteq \omega$$

Example



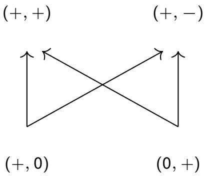
P_2

Quick Recap

- Order Complex $\Delta(P_n)$

Simplicial complex where faces are chains in P_n.

Example



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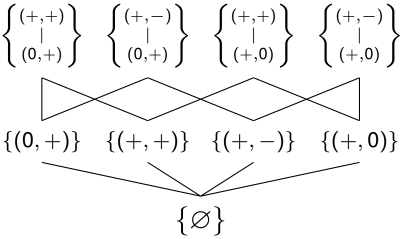


$\Delta(P_2)$

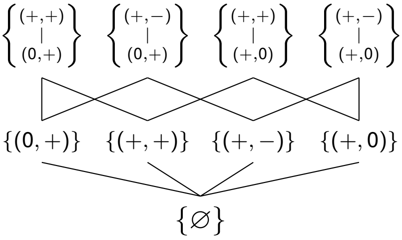
Quick Recap

- f-vector $f(\Delta) = (f_{-1}, f_0, f_1, \ldots, f_d)$

where f_i is number of i-dim faces of Δ.

Example

$$f(\Delta(P_2)) = (1, 4, 4)$$



$\Delta(P_2)$

Quick Recap

- h-vector $h(\Delta) = (h_0, h_1, \ldots, h_{d+1})$

where

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} f_{i-1}.$$

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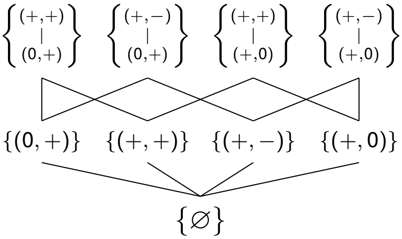

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A simplicial complex Δ is *partitionable* if

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Proposition (Stanley)

Let Δ be a partitionable simplicial complex. Then

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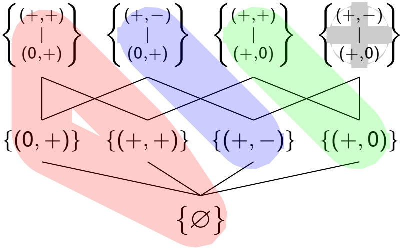

$\Delta(P_2)$

Quick Recap

Main Question: Is there a nice way to find $h(\Delta(P_n))$?

Theorem (Bergeron, D., Machacek 2020BP)

The order complex $\Delta(P_n)$ is partitionable. Moreover,

$$h_i(\Delta(P_n)) = |\{\pi \in D_n \ : \ |\mathrm{des}_B(\pi)| = i\}|$$

for each $0 \le i \le n$.

Example

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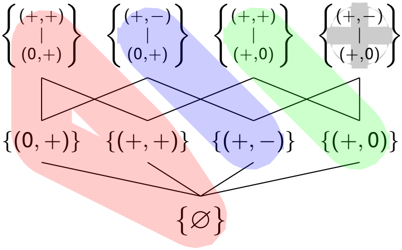

$\Delta(P_2)$

Coxeter groups

Type A_n	**Type B_n**	**Type D_n**
The elements in type A_n Coxeter groups can be represented as permutations in $\mathfrak{S}_{n+1}$.	The elements in type B_n Coxeter groups can be represented as *signed* permutations of $\mathfrak{S}_n$.	The elements in type D_n Coxeter groups can be represented as *even signed* permutations of $\mathfrak{S}_n$.
$57238146 \in A_7$	$5\bar{7}23\bar{8}\bar{1}46 \in B_8$	$5\bar{7}2\bar{3}\bar{8}\bar{1}46 \in D_8$

Descents - Type A

For $\pi = \pi_1 \dots \pi_{n+1} \in A_n$ let $\mathrm{des}_A(\pi)$ denote the descent set of π.

$$\mathrm{des}_A(\pi) = \{i \ : \ \pi_i > \pi_{i+1} \text{ for } 1 \leq i \leq n\}$$

Example

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$$12345678$$

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Example

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$$12345678$$

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$$12345678$$

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Example

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12346578

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Example

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Descents - Type B

For $\pi = \pi_1 \dots \pi_n \in B_n$ let $\mathsf{des}_B(\pi)$ denote the descent set of π.

$$\mathsf{des}_B(\pi) = \{i \ : \ \pi_i > \pi_{i+1} \text{ for } 0 \leq i < n\}$$

where $\pi_0 = 0$.

Example

$$\pi = 5\bar{7}2\bar{3}\bar{8}\bar{1}46 \in B_8$$

Descents - Type B

For $\pi = \pi_1 \dots \pi_n \in B_n$ let $\mathrm{des}_B(\pi)$ denote the descent set of π.

$$\mathrm{des}_B(\pi) = \{i \ : \ \pi_i > \pi_{i+1} \text{ for } 0 \leq i < n\}$$

where $\pi_0 = 0$.

Example

$$\pi = 5\bar{7}2\bar{3}\bar{8}\bar{1}46 \in B_8$$

To find descent, we add a 0 in front, and calculate like "normal".

$$05\bar{7}2\bar{3}\bar{8}\bar{1}46$$

$$012345678$$

$$\mathrm{des}_B(\pi) = \varnothing$$

Descents - Type B

For $\pi = \pi_1 \dots \pi_n \in B_n$ let $\mathrm{des}_B(\pi)$ denote the descent set of π.

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where $\pi_0 = 0$.

Example

$$\pi = 5\bar{7}2\bar{3}\bar{8}\bar{1}46 \in B_8$$

$$0{\color{red}5\bar{7}}2\bar{3}\bar{8}\bar{1}46$$

$$0{\color{red}12}345678$$

$$\mathrm{des}_B(\pi) = \{1\}$$

Descents - Type B

For $\pi = \pi_1 \ldots \pi_n \in B_n$ let $\mathrm{des}_B(\pi)$ denote the descent set of π.

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where $\pi_0 = 0$.

Example

$$\pi = 5\bar{7}2\bar{3}\bar{8}\bar{1}46 \in B_8$$

$$05\bar{7}2\bar{3}\bar{8}\bar{1}46$$

$$012345678$$

$$\mathrm{des}_B(\pi) = \{1, 3\}$$

Descents - Type B

For $\pi = \pi_1 \dots \pi_n \in B_n$ let $\mathrm{des}_B(\pi)$ denote the descent set of π.

$$\mathrm{des}_B(\pi) = \{i \ : \ \pi_i > \pi_{i+1} \text{ for } 0 \leq i < n\}$$

where $\pi_0 = 0$.

Example

$$\pi = 5\bar{7}2\bar{3}\bar{8}\bar{1}46 \in B_8$$

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Descents - Type B

For $\pi = \pi_1 \dots \pi_n \in B_n$ let $\mathrm{des}_B(\pi)$ denote the descent set of π.

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where $\pi_0 = 0$.

Example

$$\pi = 5\bar{7}2\bar{3}\bar{8}\bar{1}46 \in B_8$$

$$\mathrm{des}_B(\pi) = \{1, 3, 4\}$$

Descents - Type D

For $\pi = \pi_1 \ldots \pi_n \in D_n$ let $\mathrm{des}_D(\pi)$ denote the descent set of π.

$$\mathrm{des}_D(\pi) = \{i \ : \ \pi_i > \pi_{i+1} \text{ for } 0 \leq i < n\}$$

where $\pi_0 = -\pi_2$.

Example

$$\pi = 5\bar{7}2\bar{3}\bar{8}\bar{1}46 \in D_8$$

Descents - Type D

For $\pi = \pi_1 \dots \pi_n \in D_n$ let $\mathrm{des}_D(\pi)$ denote the descent set of π.

$$\mathrm{des}_D(\pi) = \{i \ : \ \pi_i > \pi_{i+1} \text{ for } 0 \leq i < n\}$$

where $\pi_0 = -\pi_2$.

Example

$$\pi = 5\bar{7}2\bar{3}\bar{8}\bar{1}46 \in D_8$$

To find descent, we add a 7 in front, and calculate like "normal".

$$75\bar{7}2\bar{3}\bar{8}\bar{1}46$$
$$012345678$$
$$\mathrm{des}_D(\pi) = \varnothing$$

Descents - Type D

For $\pi = \pi_1 \ldots \pi_n \in D_n$ let $\mathrm{des}_D(\pi)$ denote the descent set of π.

$$\mathrm{des}_D(\pi) = \{i \ : \ \pi_i > \pi_{i+1} \text{ for } 0 \leq i < n\}$$

where $\pi_0 = -\pi_2$.

Example

$$\pi = 5\bar{7}2\bar{3}\bar{8}\bar{1}46 \in D_8$$

$$75\bar{7}2\bar{3}\bar{8}\bar{1}46$$

$$012345678$$

$$\mathrm{des}_D(\pi) = \{0\}$$

Descents - Type D

For $\pi = \pi_1 \dots \pi_n \in D_n$ let $\mathrm{des}_D(\pi)$ denote the descent set of π.

$$\mathrm{des}_D(\pi) = \{i \ : \ \pi_i > \pi_{i+1} \text{ for } 0 \leq i < n\}$$

where $\pi_0 = -\pi_2$.

Example

$$\pi = 5\bar{7}2\bar{3}\bar{8}\bar{1}46 \in D_8$$

$$7 5 \bar{7} 2 \bar{3} \bar{8} \bar{1} 4 6$$

$$0 1 2 3 4 5 6 7 8$$

$$\mathrm{des}_D(\pi) = \{0, 1\}$$

Descents - Type D

For $\pi = \pi_1 \ldots \pi_n \in D_n$ let $\mathrm{des}_D(\pi)$ denote the descent set of π.

$$\mathrm{des}_D(\pi) = \{i \ : \ \pi_i > \pi_{i+1} \text{ for } 0 \leq i < n\}$$

where $\pi_0 = -\pi_2$.

Example

$$\pi = 5\bar{7}2\bar{3}\bar{8}\bar{1}46 \in D_8$$

$$75\bar{7}2\bar{3}\bar{8}\bar{1}46$$

$$012345678$$

$$\mathrm{des}_D(\pi) = \{0, 1, 3\}$$

Descents - Type D

For $\pi = \pi_1 \ldots \pi_n \in D_n$ let $\mathrm{des}_D(\pi)$ denote the descent set of π.

$$\mathrm{des}_D(\pi) = \{i \ : \ \pi_i > \pi_{i+1} \text{ for } 0 \leq i < n\}$$

where $\pi_0 = -\pi_2$.

Example

$$\pi = 5\bar{7}2\bar{3}\bar{8}\bar{1}46 \in D_8$$

$$75\bar{7}2\bar{3}\bar{8}\bar{1}46$$

$$012345678$$

$$\mathrm{des}_D(\pi) = \{0, 1, 3, 4\}$$

Descents - Type D

For $\pi = \pi_1 \dots \pi_n \in D_n$ let $\mathrm{des}_D(\pi)$ denote the descent set of π.

$$\mathrm{des}_D(\pi) = \{i \ : \ \pi_i > \pi_{i+1} \text{ for } 0 \leq i < n\}$$

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Example

$$\pi = 5\bar{7}2\bar{3}\bar{8}\bar{1}46 \in D_8$$

$$\mathrm{des}_D(\pi) = \{0, 1, 3, 4\}$$

Main Theorem

Theorem (Bergeron, D., Machacek 2020BP)

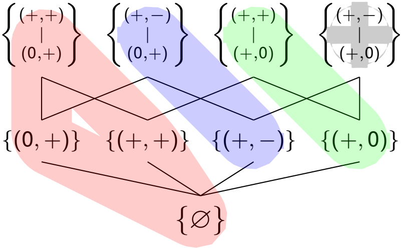
The order complex $\Delta(P_n)$ is partitionable. Moreover,

$$h_i(\Delta(P_n)) = |\{\pi \in D_n \ : \ |\mathrm{des}_B(\pi)| = i\}|$$

for each $0 \leq i \leq n$.

Example

$$h(\Delta(P_2)) = (1, 2, 1)$$



$\Delta(P_2)$

Permutations and maximal chains

How do we associate permutations and maximal chains in our poset?

- Change π_i to 0 inductively.

$57238146 \leftrightarrow$

$$(+,+,+,+,+,+,+,+)$$
$$(+,+,+,+,0,+,+,+)$$
$$(+,+,+,+,0,+,0,+)$$
$$(+,0,+,+,0,+,0,+)$$
$$(+,0,0,+,0,+,0,+)$$
$$(+,0,0,+,0,+,0,0)$$
$$(0,0,0,+,0,+,0,0)$$
$$(0,0,0,0,0,+,0,0)$$

Permutations and chains

How do we associate permutations and chains in our poset?

- Order 0s first
- Add increasing sequences inductively.

57238146

Permutations and chains

How do we associate permutations and chains in our poset?

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$57|238|146 \xrightarrow{\text{min}}$

Permutations and chains

How do we associate permutations and chains in our poset?

- Order 0s first
- Add increasing sequences inductively.

$$(+,+,+,+,0,+,0,+)$$

$$57|238|146 \xrightarrow{\text{min}}$$

Permutations and chains

How do we associate permutations and chains in our poset?

- Order 0s first
- Add increasing sequences inductively.

$$57|238|146 \xrightarrow{\text{min}} \begin{array}{c} (+,+,+,+,0,+,0,+) \\ | \\ (+,0,0,+,0,+,0,0) \end{array}$$

Permutations and chains

How do we associate permutations and chains in our poset?

- Order 0s first
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$$57|238|146 \xrightarrow{\text{min}} \begin{array}{c} (+,+,+,+,0,+,0,+) \\ | \\ (+,0,0,+,0,+,0,0) \end{array}$$

Permutations and chains

How do we associate permutations and chains in our poset?

- Order 0s first
- Add increasing sequences inductively.

$$(+,+,+,+,0,+,0,+)$$
$$|$$
$$57238146 \xrightarrow{\text{min}} \quad (+,0,0,+,0,+,0,0)$$

$$\mathrm{des}_A(\pi) = \{2,5\}$$

Permutations and chains

How do we associate permutations and chains in our poset?

- Order 0s first
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$$(+,+,+,+,0,+,0,+)$$

$$|$$

$$57238146 \xrightarrow{\text{min}} (+,0,0,+,0,+,0,0) \quad \rightarrow$$

$$\text{des}_A(\pi) = \{2,5\}$$

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- Order 0s first
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$$(+,+,+,+,0,+,0,+)$$

$$|$$

$$57238146 \xrightarrow{\text{min}} (+,0,0,+,0,+,0,0) \quad \rightarrow \quad 57$$

$$\text{des}_A(\pi) = \{2,5\}$$

Permutations and chains

How do we associate permutations and chains in our poset?

- Order 0s first
- Add increasing sequences inductively.

$$(+,+,+,+,0,+,0,+)$$

$$57238146 \xrightarrow{\text{min}} (+,0,0,+,0,+,0,0) \quad \rightarrow \quad 57238$$

$$\text{des}_A(\pi) = \{2,5\}$$

Permutations and chains

How do we associate permutations and chains in our poset?

- Order 0s first
- Add increasing sequences inductively.

$$(+,+,+,+,0,+,0,+)$$

$$|$$

$$57238146 \xrightarrow{\text{min}} \quad ({\color{red}+},0,0,{\color{red}+},0,{\color{red}+},0,0) \quad \rightarrow \quad 57238146$$

$$\mathrm{des}_A(\pi) = \{2,5\}$$

Permutations and chains

How do we associate permutations and chains in our poset?

- Order 0s first
- Add increasing sequences inductively.

$$(+,+,+,+,0,+,0,+)$$

$$|$$

$$57238146 \xrightarrow{\text{min}} (+,0,0,+,0,+,0,0) \rightarrow 57238146$$

$$\text{des}_A(\pi) = \{2,5\}$$

Negatives?

But how do we handle the negatives?!

$5\bar{7}2\bar{3}\bar{8}\bar{1}46 \quad \leftrightarrow \quad ?$

Negatives?

But how do we handle the negatives?!

$5\bar{7}2\bar{3}\bar{8}\bar{1}46 \quad \leftrightarrow \quad ?$

$(57238146, \{1, 3, 7, 8\})$

Sign Variations

Sign vector $\omega \in \mathcal{V}_n = \{+, 0, -\}^n$.

$$\text{var}(\omega) = \text{number of times } \omega \text{ changes sign}$$

$i \in [n]$ is a *sign flip* of ω if there exists a j such that $\omega_{i-j}\omega_i < 0$ while $\omega_{i-k}\omega_i = 0$ for all $1 \leq k < j$.

Example

$$\omega = (+, +, -, -, -, -, +, -) \Rightarrow \text{var}(\omega) = 3$$

$$\begin{array}{c} (+, +, -, -, -, 0, +, -) \\ 1 \ \ 2 \ \ 3 \ \ 4 \ \ 5 \ \ 6 \ \ 7 \ \ 8 \end{array} \quad \leftrightarrow \quad \{3, 7, 8\}$$

Cyclic Sign Variations

Sign vector $\omega \in \mathcal{V}_n = \{+, 0, -\}^n$.

$$\mathrm{cvar}(\omega) = \text{number of times } \omega \text{ changes sign, cyclically}$$

$i \in [n]$ is a *cyclic sign flip* of ω if there exists a j such that $\omega_{i-j}\omega_i < 0$ while $\omega_{i-k}\omega_i = 0$ for all $1 \leq k < j$ where $\omega_i = \omega_{i+n}$.

Example

$$\omega = (+, +, -, -, -, -, +, -) \Rightarrow \mathrm{cvar}(\omega) = 4$$

$$\begin{array}{c} (+, +, -, -, -, 0, +, -) \\ 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7\ \ 8 \end{array} \quad \leftrightarrow \quad \{1, 3, 7, 8\}$$

Even signed permutations and maximal chains

How about even signed permutations and maximal chains?

- Just as before with a final step of adding negatives.

$$5\bar{7}2\bar{3}\bar{8}\bar{1}46 \leftrightarrow (57238146, \{1,3,7,8\})$$

$$(+,+,-,-,-,-,+,-)$$
$$(+,+,-,-,0,-,+,-)$$
$$(+,+,-,-,0,-,0,-)$$
$$(+,0,-,-,0,-,0,-)$$

$$(+,0,0,-,0,-,0,-)$$
$$(+,0,0,-,0,-,0,0)$$
$$(0,0,0,+,0,+,0,0)$$
$$(0,0,0,0,0,+,0,0)$$

Even signed permutations and chains

How do we associate even signed permutations and chains in our poset?

- Just as before, but with negatives.

$$5\bar{7}2\bar{3}\bar{8}\bar{1}46$$

$$(57238146, \{1, 3, 7, 8\})$$

Even signed permutations and chains

How do we associate even signed permutations and chains in our poset?

- Just as before, but with negatives.

$$5|\bar{7}2|\bar{3}|\bar{8}\bar{1}46 \xrightarrow{\min}$$

$$(57238146, \{1, 3, 7, 8\})$$

Even signed permutations and chains

How do we associate even signed permutations and chains in our poset?

- Just as before, but with negatives.

$(+,+,-,-,-,-,+,-)$

$5|\bar{7}2|\bar{3}|\bar{8}\bar{1}46 \xrightarrow{\text{min}}$

$(57238146, \{1,3,7,8\})$

Even signed permutations and chains

How do we associate even signed permutations and chains in our poset?

- Just as before, but with negatives.

$$(+,+,-,-,-,-,+,-)$$

$$(+,+,-,-,0,-,+,-)$$

$$5|\bar{7}2|\bar{3}|\bar{8}\bar{1}46 \xrightarrow{\text{min}}$$

$$(57238146, \{1,3,7,8\})$$

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$$(+,+,-,-,-,-,+,-)$$

$$(+,+,-,-,0,-,+,-)$$

$$5|\bar{7}2|\bar{3}|\bar{8}\bar{1}46 \xrightarrow{\text{min}}$$

$$(57238146,\{1,3,7,8\}) \qquad (+,0,-,-,0,-,0,-)$$

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$$(+,+,-,-,-,-,+,-)$$

$$(+,+,-,-,0,-,+,-)$$

$$5|\bar{7}2|\bar{3}|\bar{8}\bar{1}46 \xrightarrow{\text{min}} \qquad \Big|$$

$$(57238146, \{1,3,7,8\}) \qquad (+,0,-,-,0,-,0,-)$$

$$\Big|$$

$$(+,0,0,-,0,-,0,-)$$

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$$(+,+,-,-,-,-,+,-)$$

$$(+,+,-,-,0,-,+,-)$$

$$5|\bar{7}2|\bar{3}|\bar{8}\bar{1}46 \xrightarrow{\text{min}} \qquad (+,0,-,-,0,-,0,-)$$

$$(57238146, \{1,3,7,8\})$$

$$(+,0,0,-,0,-,0,-)$$

Even signed permutations and chains

How do we associate even signed permutations and chains in our poset?

- Just as before, but with negatives.

$$(+,+,-,-,-,-,+,-)$$

$$5|\bar{7}2|\bar{3}|\bar{8}\bar{1}46 \xrightarrow{\text{min}} (+,+,-,-,0,-,+,-)$$

$$(57238146, \{1,3,7,8\})$$

$$(+,0,-,-,0,-,0,-)$$

$$(+,0,0,-,0,-,0,-)$$

$$\mathrm{des}_D(\pi) = \{0,1,3,4\}$$

Even signed permutations and chains

How do we associate even signed permutations and chains in our poset?

- Just as before, but with negatives.

$$(+,+,-,-,-,-,+,-)$$

$$5|\bar{7}2|\bar{3}|\bar{8}\bar{1}46 \xrightarrow{\text{min}} (+,+,-,-,0,-,+,-)$$

$$(57238146,\{1,3,7,8\}) \qquad (+,0,-,-,0,-,0,-)$$

$$(+,0,0,-,0,-,0,-)$$

~~$\text{des}_D(\pi)=\{0,1,3,4\}$~~

$$\text{des}_B(\pi)=\{1,3,4\}$$

Even signed permutations and chains

How do we associate even signed permutations and chains in our poset?

- Just as before, but with negatives.

$$5\bar{7}2\bar{3}\bar{8}\bar{1}46 \xrightarrow{\text{min}} \begin{array}{c} (+,+,-,-,0,-,+,-) \\ | \\ (+,0,-,-,0,-,0,-) \\ | \\ (+,0,0,-,0,-,0,-) \end{array} \rightarrow$$

Even signed permutations and chains

How do we associate even signed permutations and chains in our poset?

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$$5\bar{7}2\bar{3}\bar{8}\bar{1}46 \xrightarrow{\text{min}} \begin{array}{c} (+,+,-,-,0,-,+,-) \\ | \\ (+,0,-,-,0,-,0,-) \\ | \\ (+,0,0,-,0,-,0,-) \end{array} \rightarrow \{1,3,7,8\}$$

Even signed permutations and chains

How do we associate even signed permutations and chains in our poset?

- Just as before, but with negatives.

$$5\bar{7}2\bar{3}\bar{8}\bar{1}46 \xrightarrow{\text{min}} \begin{array}{c} (+,+,-,-,{\color{red}0},-,+,-) \\ | \\ (+,0,-,-,0,-,0,-) \\ | \\ (+,0,0,-,0,-,0,-) \end{array} \rightarrow \begin{array}{c} \{1,3,7,8\} \\ \\ 5 \end{array}$$

Even signed permutations and chains

How do we associate even signed permutations and chains in our poset?

- Just as before, but with negatives.

$$(+,+,-,-,0,-,+,-)$$

$$5\bar{7}2\bar{3}\bar{8}\bar{1}46 \xrightarrow{\min} (+,0,-,-,0,-,0,-) \rightarrow$$

$$(+,0,0,-,0,-,0,-)$$

$$\{1,3,7,8\}$$

$$5\bar{7}2$$

Even signed permutations and chains

How do we associate even signed permutations and chains in our poset?

- Just as before, but with negatives.

$$
5\bar{7}2\bar{3}\bar{8}\bar{1}46 \xrightarrow{\text{min}}
\begin{array}{c}
(+,+,-,-,0,-,+,-) \\
| \\
(+,0,-,-,0,-,0,-) \\
| \\
(+,0,{\color{red}0},-,0,-,0,-)
\end{array}
\rightarrow
\begin{array}{c}
\{1,3,7,8\} \\
\\
5\bar{7}2\bar{3}
\end{array}
$$

Even signed permutations and chains

How do we associate even signed permutations and chains in our poset?

- Just as before, but with negatives.

$$(+,+,-,-,0,-,+,-)$$

$$(+,0,-,-,0,-,0,-) \quad \rightarrow \qquad \{1,3,7,8\}$$

$$5\bar{7}2\bar{3}\bar{8}\bar{1}46 \xrightarrow{\text{min}}$$

$$(+,0,0,-,0,-,0,-) \qquad\qquad 5\bar{7}2\bar{3}\bar{8}\bar{1}46$$

Even signed permutations and chains

How do we associate even signed permutations and chains in our poset?

- Just as before, but with negatives.

$$
5\bar{7}2\bar{3}\bar{8}\bar{1}46 \xrightarrow{\text{min}} \begin{array}{c} (+,+,-,-,0,-,+,-) \\ | \\ (+,0,-,-,0,-,0,-) \\ | \\ (+,0,0,-,0,-,0,-) \end{array} \rightarrow \qquad 5\bar{7}2\bar{3}\bar{8}\bar{1}46
$$

Main Theorem (Again)

Let D_n be the type D Coxeter group and let des_B denote the type B descent set of an element $\pi \in D_n$.

Theorem (Bergeron, D., Machacek 2020BP)

The order complex $\Delta(P_n)$ is partitionable. Moreover,

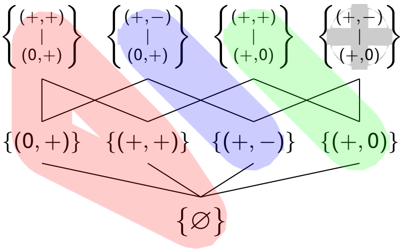
$$h_i(\Delta(P_n)) = |\{\pi \in D_n \ : \ |\mathrm{des}_B(\pi)| = i\}|$$

for each $0 \leq i \leq n$.

Example - $n = 2$

$$h_i(\Delta(P_2)) = |\{\pi \in D_2 \; : \; |\mathrm{des}_B(\pi)| = i\}| \qquad h(\Delta(P_2)) = (1, 2, 1)$$

D_n	des_B
12	$\varnothing$
$\bar{2}\bar{1}$	$\{0\}$
21	$\{1\}$
$\bar{1}\bar{2}$	$\{0,1\}$



$\Delta(P_2)$

Restriction of variations

- $\mathcal{PV}_{n,m} = \{\omega \in \mathcal{PV}_n \ : \ \mathrm{var}(\omega) \leq m\}$.
- $P_{n,m} = (\mathcal{PV}_{n,m}, <)$.
- $D_{n,m} = \{\pi \in D_n \ : \ \pi$ has at most m negatives$\}$.

Theorem (Bergeron, D., Machacek 2020BP)

If $m \leq n-1$ is even then the order complex $\Delta(P_{n,m})$ is partitionable. Moreover,

$$h_i(\Delta(P_{n,m})) = |\{\pi \in D_{n,m} \ : \ |\mathrm{des}_B(\pi)| = i\}|$$

for each $0 \leq i \leq n$.

Thank you!

$\Delta(P_3)$