The facial weak order and its lattice of quotients

Aram Dermenjian

Joint work with: Christophe Hohlweg (LACIM) and Vincent Pilaud (CNRS & LIX)

Université du Québec à Montréal

13 April 2019

On this day in 1909 Stan Ulam was born.

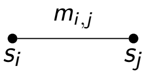
"Knowing what is big and what is small is more important than being able to solve partial differential equations." - Ulam.

History and Background

- *Finite Coxeter System* (W, S) such that

$$W := \langle s \in S \mid (s_i s_j)^{m_{i,j}} = e \text{ for } s_i, s_j \in S \rangle$$

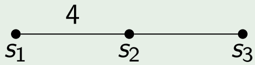
  where $m_{i,j} \in \mathbb{N}^\star$ and $m_{i,j} = 1$ only if $i = j$.
- A *Coxeter diagram* Γ_W for a Coxeter System (W, S) has S as a vertex set and an edge labelled $m_{i,j}$ when $m_{i,j} > 2$.



Example

$$W_{B_3} = \left\langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^4 = (s_2 s_3)^3 = (s_1 s_3)^2 = e \right\rangle$$

Γ_{B_3} :



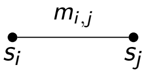
History and Background

- *Finite Coxeter System* (W, S) such that

$$W := \langle s \in S \mid (s_i s_j)^{m_{i,j}} = e \text{ for } s_i, s_j \in S \rangle$$

where $m_{i,j} \in \mathbb{N}^\star$ and $m_{i,j} = 1$ only if $i = j$.

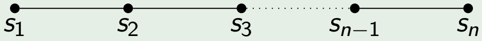
- A *Coxeter diagram* Γ_W for a Coxeter System (W, S) has S as a vertex set and an edge labelled $m_{i,j}$ when $m_{i,j} > 2$.



Example

$W_{A_n} = S_{n+1}$, symmetric group.

Γ_{A_n} :



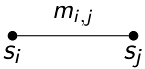
History and Background

- *Finite Coxeter System* (W, S) such that

$$W := \langle s \in S \mid (s_i s_j)^{m_{i,j}} = e \text{ for } s_i, s_j \in S \rangle$$

where $m_{i,j} \in \mathbb{N}^\star$ and $m_{i,j} = 1$ only if $i = j$.

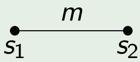
- A *Coxeter diagram* Γ_W for a Coxeter System (W, S) has S as a vertex set and an edge labelled $m_{i,j}$ when $m_{i,j} > 2$.



Example

$W_{I_2(m)} = \mathcal{D}(m)$, dihedral group of order $2m$.

$\Gamma_{I_2(m)}$:



History and Background

Let (W, S) be a Coxeter system.

- Let $w \in W$ such that $w = s_1 \dots s_n$ for some $s_i \in S$. We say that w has *length* n, $\ell(w) = n$, if n is minimal.

Example

Let Γ_{A_2} : $\overset{s}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{t}{\bullet}$.

$\ell(stst) = 2$ as $stst = tstt = ts$.

- Let the *(right) weak order* be the order on the Cayley graph where $\overset{w}{\bullet} \longrightarrow \overset{ws}{\bullet}$ and $\ell(w) < \ell(ws)$.

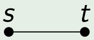
History and Background

Theorem (Björner [1984])

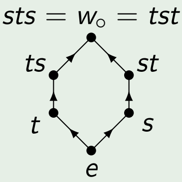
Let (W, S) be a finite Coxeter system. The weak order is a lattice graded by length.

- For finite Coxeter systems, there exists a longest element in the weak order, $w_\circ$.

Example

Let Γ_{A_2} : 

.



Motivation

- In 2001, Krob, Latapy, Novelli, Phan, and Schwer extended the weak order to an order on all faces for type A using inversion tables. They
  1. gave a local definition of this order using covers,
  2. gave a global definition of this order combinatorially, and
  3. showed that the poset for this order is a lattice.
- In 2006, Ronco and Palacios extended this new order to Coxeter groups of all types using cover relations.

Motivation

- In 2001, Krob, Latapy, Novelli, Phan, and Schwer extended the weak order to an order on all faces for type A using inversion tables. They
  1. gave a local definition of this order using covers, ✓
  2. gave a global definition of this order combinatorially, and
  3. showed that the poset for this order is a lattice.
- In 2006, Ronco and Palacios extended this new order to Coxeter groups of all types using cover relations.
- Problems: Can we find a global definition for this poset, and is it a lattice?

Motivation

- In 2001, Krob, Latapy, Novelli, Phan, and Schwer extended the weak order to an order on all faces for type A using inversion tables. They
  1. gave a local definition of this order using covers, ✓
  2. gave a global definition of this order combinatorially, and ✓
  3. showed that the poset for this order is a lattice. ✓
- In 2006, Ronco and Palacios extended this new order to Coxeter groups of all types using cover relations.
- Problems: Can we find a global definition for this poset, and is it a lattice?

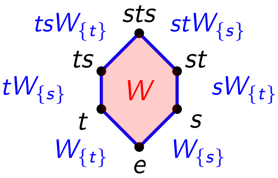
Motivation

- In 2001, Krob, Latapy, Novelli, Phan, and Schwer extended the weak order to an order on all faces for type A using inversion tables. They
  1. gave a local definition of this order using covers, ✓
  2. gave a global definition of this order combinatorially, and ✓
  3. showed that the poset for this order is a lattice. ✓
- In 2006, Ronco and Palacios extended this new order to Coxeter groups of all types using cover relations.
- Problems: Can we find a global definition for this poset, and is it a lattice?

Parabolic Subgroups

Let $I \subseteq S$.

- $W_I = \langle I \rangle$ is *standard parabolic subgroup* (long element: $w_{\circ,I}$).
- $W^I := \{w \in W \mid \ell(w) \leq \ell(ws), \text{ for all } s \in I\}$ is the set of min length coset representatives for W/W_I.
- Unique factorization: $w = w^I \cdot w_I$ with $w^I \in W^I$, $w_I \in W_I$.
- By convention in this talk xW_I means $x \in W^I$.
- *Coxeter complex* - $\mathcal{P}_W$ - the abstract simplicial complex whose faces are all the standard parabolic cosets of W.



Facial Weak Order

Let (W, S) be a finite Coxeter system.

Definition (Krob et.al. [2001, type A], Palacios, Ronco [2006])

The *(right) facial weak order* is the order $\leq_F$ on the Coxeter complex $\mathcal{P}_W$ defined by cover relations of two types:

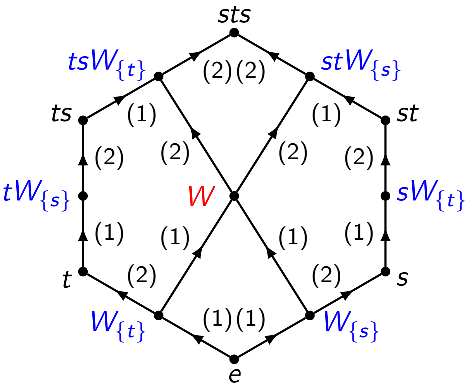
$$(1) \qquad xW_I \lessdot xW_{I\cup\{s\}} \qquad \text{if } s \notin I \text{ and } x \in W^{I\cup\{s\}},$$

$$(2) \qquad xW_I \lessdot xw_{\circ,I}w_{\circ,I\smallsetminus\{s\}}W_{I\smallsetminus\{s\}} \qquad \text{if } s \in I,$$

where $I \subseteq S$ and $x \in W^I$.

Facial weak order example

(1) $xW_I \lessdot xW_{I\cup\{s\}}$ if $s \notin I$ and $x \in W^{I\cup\{s\}}$
(2) $xW_I \lessdot xw_{\circ,I}w_{\circ,I\smallsetminus\{s\}}W_{I\smallsetminus\{s\}}$ if $s \in I$



Motivation

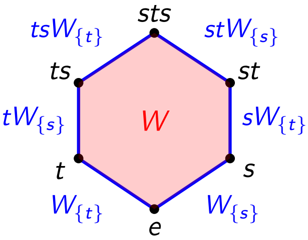
- In 2001, Krob, Latapy, Novelli, Phan, and Schwer extended the weak order to an order on all faces for type A using inversion tables. They
  1. gave a local definition of this order using covers, ✓
  2. gave a global definition of this order combinatorially, and ✓
  3. showed that the poset for this order is a lattice. ✓
- In 2006, Ronco and Palacios extended this new order to Coxeter groups of all types using cover relations.
- Problems: Can we find a global definition for this poset, and is it a lattice?

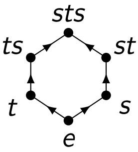
Facial Intervals

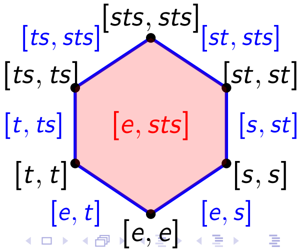
Proposition (Björner, Las Vergas, Sturmfels, White, Ziegler '93)

Let (W, S) be a finite Coxeter system and xW_I a standard parabolic coset. Then there exists a unique interval $[x, xw_{\circ,I}]$ in the weak order such that

$$xW_I = [x, xw_{\circ,I}].$$





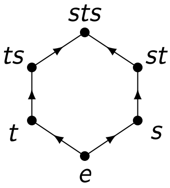


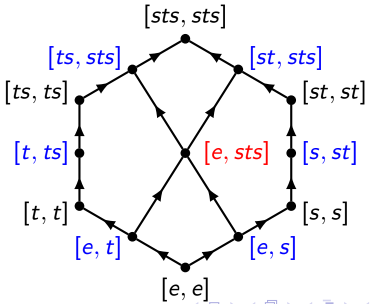
Global Definition

Definition

Let $\leq_{F'}$ be the order on the Coxeter complex $\mathcal{P}_W$ defined by

$$xW_I \leq_{F'} yW_J \Leftrightarrow x \leq_R y \text{ and } xw_{\circ,I} \leq_R yw_{\circ,J}$$

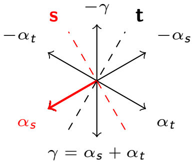




Root System

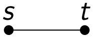
- Let $(V, \langle\cdot,\cdot\rangle)$ be a real Euclidean space.
- Let W be a group generated by a set of reflections S.
  $W \hookrightarrow O(V)$ gives representation as a finite reflection group.
- The reflection associated to $\alpha \in V\backslash\{0\}$ is

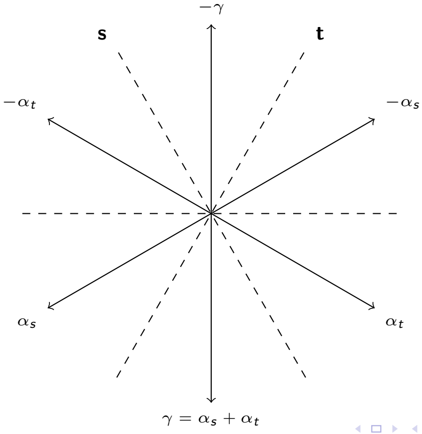
$$s_\alpha(v) = v - \frac{2\langle v, \alpha\rangle}{||\alpha||^2}\alpha \quad (v \in V)$$



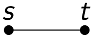
- A *root system* is $\Phi := \{\alpha \in V \mid s_\alpha \in W, ||\alpha|| = 1\}$
- We have $\Phi = \Phi^+ \sqcup \Phi^-$ decomposable into positive and negative roots.

Relationship between Root Systems and Coxeter Systems

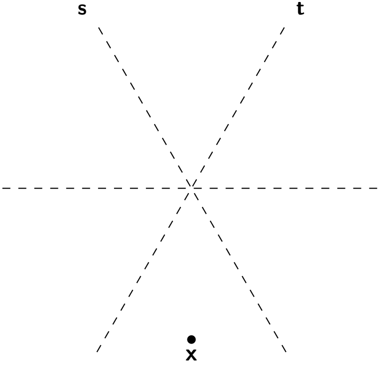
$W_{A_2} = \langle s, t \mid s^2 = t^2 = (st)^3 = e \rangle$ Γ_{A_2} : 



Relationship between Root Systems and Coxeter Systems

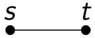
$W_{A_2} = \langle s, t \mid s^2 = t^2 = (st)^3 = e \rangle$ Γ_{A_2} : 

$\mathrm{Perm}(W) = \{w(x) \mid w \in W\}$

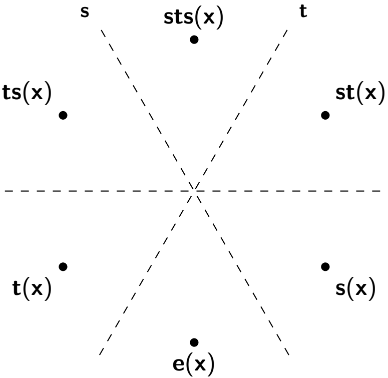


Relationship between Root Systems and Coxeter Systems

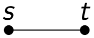
$W_{A_2} = \langle s, t \mid s^2 = t^2 = (st)^3 = e \rangle$ Γ_{A_2} :



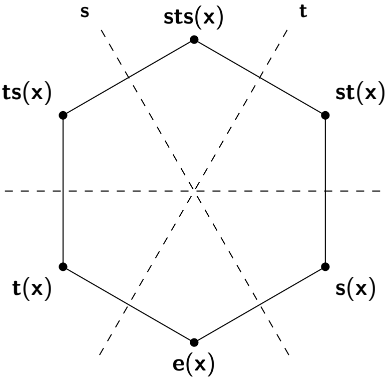
$\mathrm{Perm}(W) = \{w(x) \mid w \in W\}$



Relationship between Root Systems and Coxeter Systems

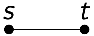
$W_{A_2} = \langle s, t \mid s^2 = t^2 = (st)^3 = e \rangle$ Γ_{A_2} : 

$\mathrm{Perm}(W) = \{w(x) \mid w \in W\}$

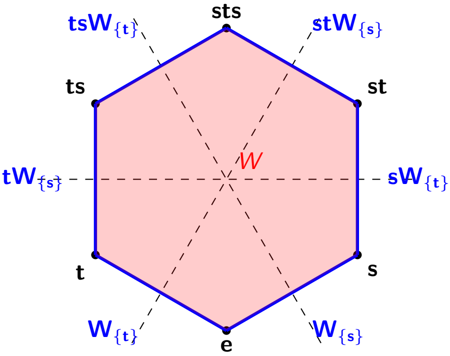


Relationship between Root Systems and Coxeter Systems

$W_{A_2} = \langle s, t \mid s^2 = t^2 = (st)^3 = e \rangle$ Γ_{A_2} :



$\mathrm{Perm}(W) = \{w(x) \mid w \in W\}$

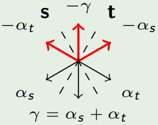


Inversion Sets

Let (W, S) be a Coxeter system.
Define *(left) inversion sets* as the set $\mathbf{N}(w) := \Phi^+ \cap w(\Phi^-)$.

Example

Let Γ_{A_2} : s ●——● t , with Φ given by the roots



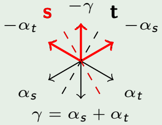
$$\begin{aligned}\mathbf{N}(ts) &= \Phi^+ \cap ts(\Phi^-) \\ &= \Phi^+ \cap \{\alpha_t, \gamma, -\alpha_s\} \\ &= \{\alpha_t, \gamma\}\end{aligned}$$

Inversion Sets

Let (W, S) be a Coxeter system.
Define *(left) inversion sets* as the set $\mathbf{N}(w) := \Phi^+ \cap w(\Phi^-)$.

Example

Let Γ_{A_2} : s ——— t , with Φ given by the roots



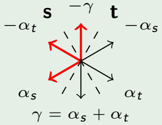
$$\begin{aligned}\mathbf{N}(ts) &= \Phi^+ \cap ts(\Phi^-) \\ &= \Phi^+ \cap \{\alpha_t, \gamma, -\alpha_s\} \\ &= \{\alpha_t, \gamma\}\end{aligned}$$

Inversion Sets

Let (W, S) be a Coxeter system.
Define *(left) inversion sets* as the set $\mathbf{N}(w) := \Phi^+ \cap w(\Phi^-)$.

Example

Let Γ_{A_2} : $\overset{s}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{t}{\bullet}$, with Φ given by the roots



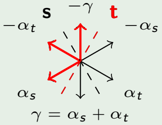
$$\begin{aligned}\mathbf{N}(ts) &= \Phi^+ \cap ts(\Phi^-)\\ &= \Phi^+ \cap \{\alpha_t, \gamma, -\alpha_s\}\\ &= \{\alpha_t, \gamma\}\end{aligned}$$

Inversion Sets

Let (W, S) be a Coxeter system.
Define *(left) inversion sets* as the set $\mathbf{N}(w) := \Phi^+ \cap w(\Phi^-)$.

Example

Let Γ_{A_2} : $\overset{s}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{t}{\bullet}$, with Φ given by the roots



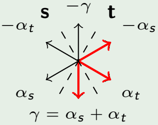
$$\begin{aligned}\mathbf{N}(ts) &= \Phi^+ \cap ts(\Phi^-)\\ &= \Phi^+ \cap \{\alpha_t, \gamma, -\alpha_s\}\\ &= \{\alpha_t, \gamma\}\end{aligned}$$

Inversion Sets

Let (W, S) be a Coxeter system.
Define *(left) inversion sets* as the set $\mathbf{N}(w) := \Phi^+ \cap w(\Phi^-)$.

Example

Let Γ_{A_2} : $\overset{s}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{t}{\bullet}$, with Φ given by the roots

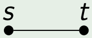


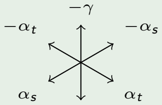
$$
\begin{aligned}
\mathbf{N}(ts) &= \Phi^+ \cap ts(\Phi^-) \\
&= \Phi^+ \cap \{\alpha_t, \gamma, -\alpha_s\} \\
&= \{\alpha_t, \gamma\}
\end{aligned}
$$

Weak order and Inversion sets

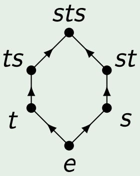
Given $w, u \in W$ then $w \leq_R u$ if and only if $\mathbf{N}(w) \subseteq \mathbf{N}(u)$.

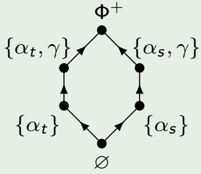
Example

Let Γ_{A_2} : 

, with Φ given by the roots 

$\gamma = \alpha_s + \alpha_t$





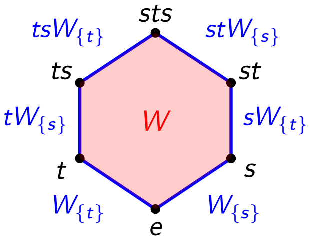
Root Inversion Set

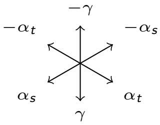
Definition (Root Inversion Set)

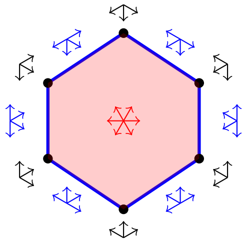
Let xW_I be a standard parabolic coset. The *root inversion set* is the set

$$\mathbf{R}(xW_I) := x(\Phi^- \cup \Phi_I^+)$$

Note that $N(x) = \mathbf{R}(xW_\varnothing) \cap \Phi^+$.



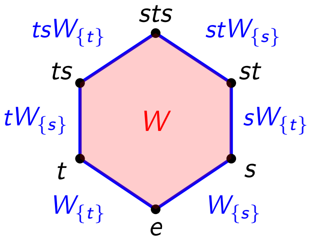


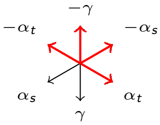


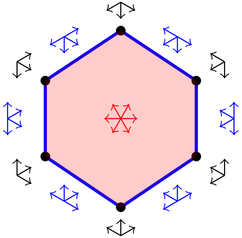
Root Inversion Set

Example

$$\begin{aligned}\mathbf{R}(sW_{\{t\}}) &= s(\Phi^- \cup \Phi^+_{\{t\}})\\ &= s(\{-\alpha_s, -\alpha_t, -\gamma\} \cup \{\alpha_t\})\\ &= \{\alpha_s, -\gamma, -\alpha_t, \gamma\}\end{aligned}$$



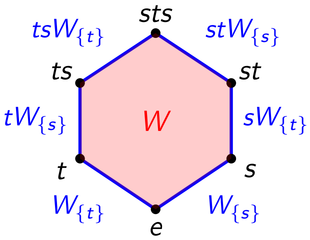


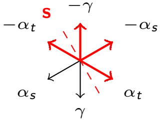


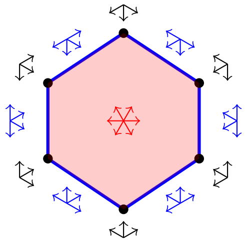
Root Inversion Set

Example

$$\begin{aligned}\mathbf{R}(sW_{\{t\}}) &= s(\Phi^{-} \cup \Phi^{+}_{\{t\}})\\ &= s(\{-\alpha_s, -\alpha_t, -\gamma\} \cup \{\alpha_t\})\\ &= \{\alpha_s, -\gamma, -\alpha_t, \gamma\}\end{aligned}$$



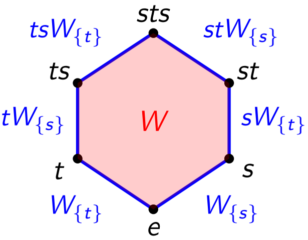


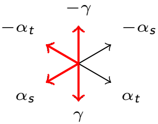


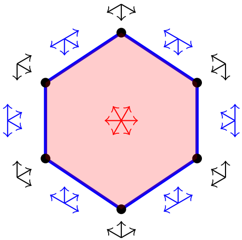
Root Inversion Set

Example

$$\begin{aligned}\mathbf{R}(sW_{\{t\}}) &= s(\Phi^- \cup \Phi^+_{\{t\}}) \\ &= s(\{-\alpha_s, -\alpha_t, -\gamma\} \cup \{\alpha_t\}) \\ &= \{\alpha_s, -\gamma, -\alpha_t, \gamma\}\end{aligned}$$



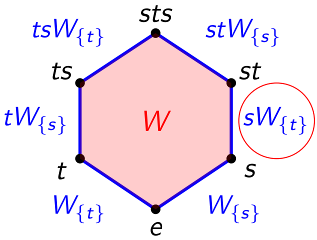


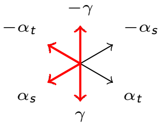


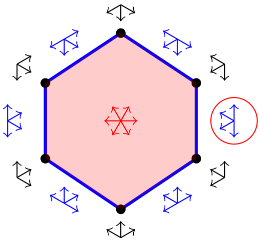
Root Inversion Set

Example

$$\begin{aligned}\mathbf{R}(sW_{\{t\}}) &= s(\Phi^- \cup \Phi^+_{\{t\}}) \\ &= s(\{-\alpha_s, -\alpha_t, -\gamma\} \cup \{\alpha_t\}) \\ &= \{\alpha_s, -\gamma, -\alpha_t, \gamma\}\end{aligned}$$







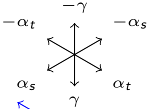
Equivalent definitions

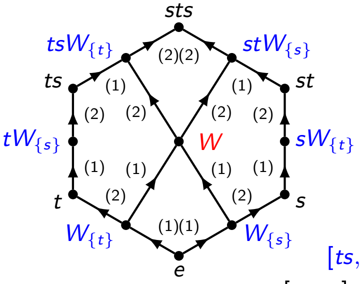
Theorem (D., Hohlweg, Pilaud [2016])

The following conditions are equivalent for two standard parabolic cosets xW_I and yW_J in the Coxeter complex $\mathcal{P}_W$

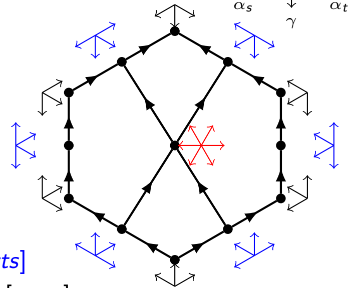
1. $xW_I \leq_F yW_J$
2. $\mathbf{R}(xW_I) \smallsetminus \mathbf{R}(yW_J) \subseteq \Phi^-$ *and* $\mathbf{R}(yW_J) \smallsetminus \mathbf{R}(xW_I) \subseteq \Phi^+$.
3. $x \leq_R y$ *and* $xw_{\circ,I} \leq_R yw_{\circ,J}$.

Equivalence for type A_2 Coxeter System



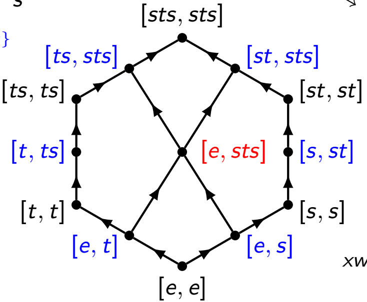


$xW_I \leq_F yW_J$



$\mathbf{R}(xW_I) \smallsetminus \mathbf{R}(yW_J) \subseteq \Phi^-$

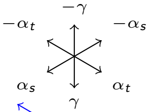
$\mathbf{R}(yW_J) \smallsetminus \mathbf{R}(xW_I) \subseteq \Phi^+$

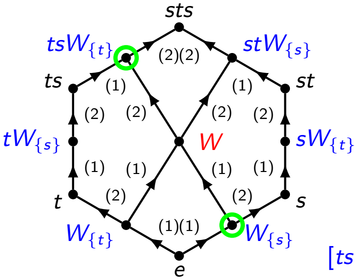


$x \leq_R y$

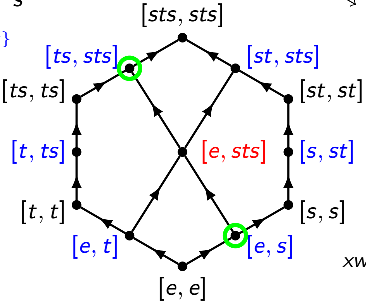
$xw_{\circ,I} \leq_R yw_{\circ,J}$

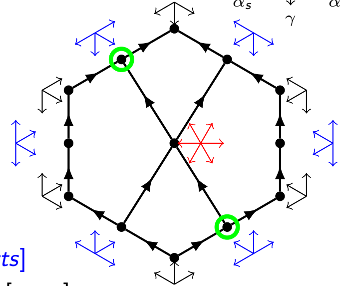
Equivalence for type A_2 Coxeter System





$xW_I \leq_F yW_J$





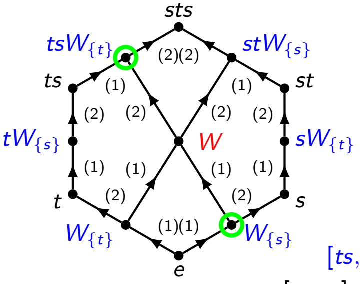
$\mathbf{R}(xW_I) \smallsetminus \mathbf{R}(yW_J) \subseteq \Phi^-$

$\mathbf{R}(yW_J) \smallsetminus \mathbf{R}(xW_I) \subseteq \Phi^+$

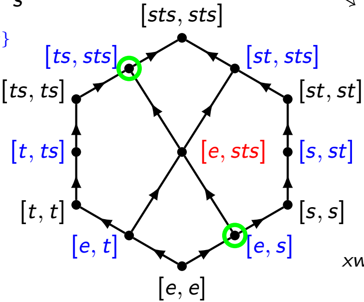
$x \leq_R y$

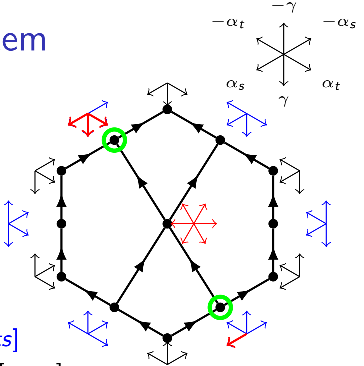
$xw_{\circ,I} \leq_R yw_{\circ,J}$

Equivalence for type A_2 Coxeter System



$xW_I \leq_F yW_J$





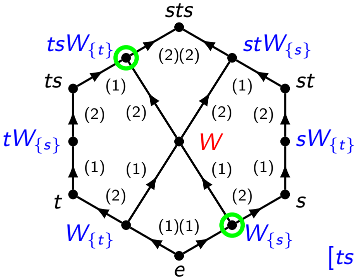
$\mathbf{R}(xW_I) \smallsetminus \mathbf{R}(yW_J) \subseteq \Phi^-$

$\mathbf{R}(yW_J) \smallsetminus \mathbf{R}(xW_I) \subseteq \Phi^+$

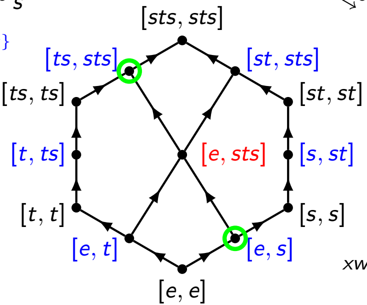
$x \leq_R y$

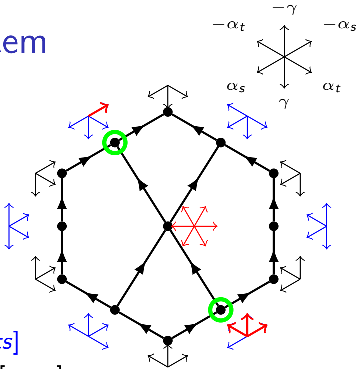
$xw_{\circ,I} \leq_R yw_{\circ,J}$

Equivalence for type A_2 Coxeter System



$xW_I \leq_F yW_J$





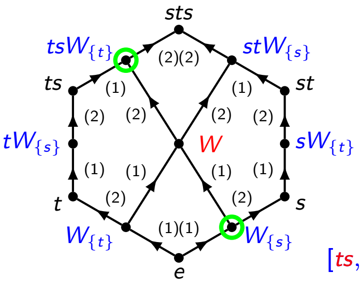
$\mathbf{R}(xW_I) \smallsetminus \mathbf{R}(yW_J) \subseteq \Phi^-$

$\mathbf{R}(yW_J) \smallsetminus \mathbf{R}(xW_I) \subseteq \Phi^+$

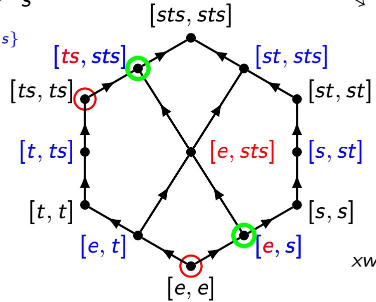
$x \leq_R y$

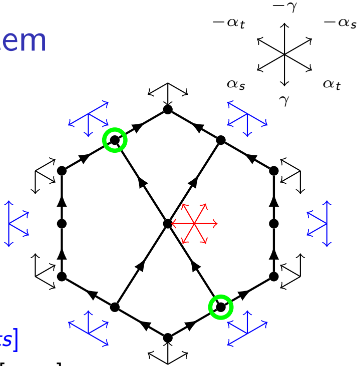
$xw_{\circ,I} \leq_R yw_{\circ,J}$

Equivalence for type A_2 Coxeter System



$xW_I \leq_F yW_J$





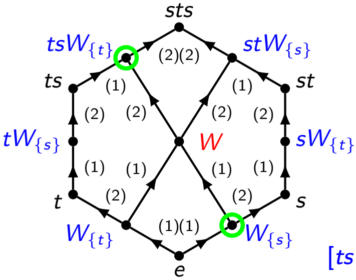
$\mathbf{R}(xW_I) \smallsetminus \mathbf{R}(yW_J) \subseteq \Phi^-$

$\mathbf{R}(yW_J) \smallsetminus \mathbf{R}(xW_I) \subseteq \Phi^+$

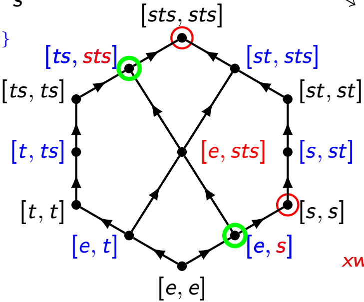
$x \leq_R y$

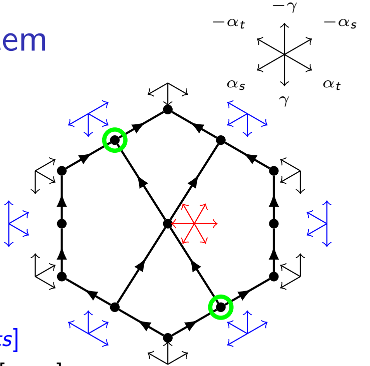
$xw_{\circ,I} \leq_R yw_{\circ,J}$

Equivalence for type A_2 Coxeter System



$xW_I \leq_F yW_J$





$\mathbf{R}(xW_I) \smallsetminus \mathbf{R}(yW_J) \subseteq \Phi^-$

$\mathbf{R}(yW_J) \smallsetminus \mathbf{R}(xW_I) \subseteq \Phi^+$

$x \leq_R y$

$xw_{\circ,I} \leq_R yw_{\circ,J}$

Motivation

- In 2001, Krob, Latapy, Novelli, Phan, and Schwer extended the weak order to an order on all faces for type A using inversion tables. They
  1. gave a local definition of this order using covers, ✓
  2. gave a global definition of this order combinatorially, and ✓
  3. showed that the poset for this order is a lattice. ✓
- In 2006, Ronco and Palacios extended this new order to Coxeter groups of all types using cover relations.
- Problems: Can we find a global definition for this poset, and is it a lattice?

Facial weak order lattice

Theorem (D., Hohlweg, Pilaud [2016])

The facial weak order $(\mathcal{P}_W, \leq_F)$ *is a lattice with the meet and join of two standard parabolic cosets* xW_I *and* yW_J *given by:*

$$xW_I \wedge yW_J = z_\wedge W_{K_\wedge},$$
$$xW_I \vee yW_J = z_\vee W_{K_\vee}.$$

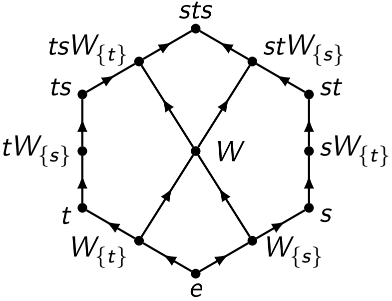
where,

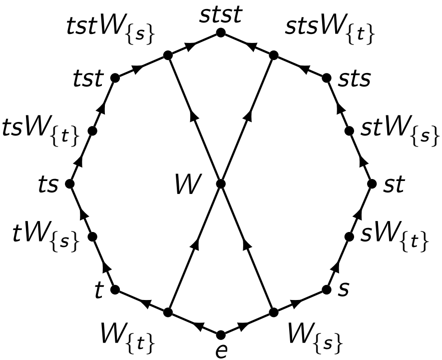
$$z_\wedge = x \wedge y \quad \text{and} \quad K_\wedge = D_L\big(z_\wedge^{-1}(xw_{\circ,I} \wedge yw_{\circ,J})\big), \text{ and}$$
$$z_\vee = xw_{\circ,I} \vee yw_{\circ,J} \quad \text{and} \quad K_\vee = D_L\big(z_\vee^{-1}(x \vee y)\big)$$

Corollary (D., Hohlweg, Pilaud [2016])

The weak order is a sublattice of the facial weak order lattice.

Example: A_2 and B_2





Example: A_2 and B_2

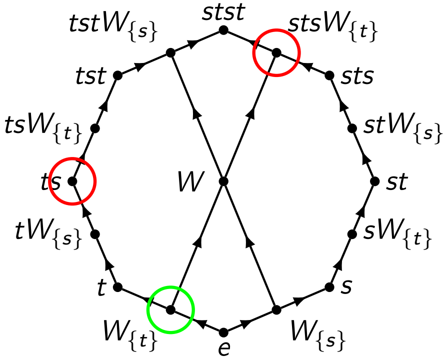
Example (Meet example)

Recall

$$xW_I \wedge yW_J = z_\wedge W_{K_\wedge}$$
$$\text{where} \quad z_\wedge = x \wedge y$$
$$K_\wedge = D_L(z_\wedge^{-1}(xw_{\circ,I} \wedge yw_{\circ,J}))$$

We compute $ts \wedge stsW_{\{t\}}$.

$$z_\wedge = ts \wedge sts = e$$
$$K_\wedge = D_L(z_\wedge^{-1}(tsw_{\circ,\emptyset} \wedge stsw_{\circ,t}))$$
$$= D_L(e(ts \wedge stst))$$
$$= D_L(ts) = \{t\}.$$



Möbius function

Recall that the *Möbius function* of a poset $(P, \leq)$ is the function $\mu : P \times P \to \mathbb{Z}$ defined inductively by

$$\mu(p,q) := \begin{cases} 1 & \text{if } p = q, \\ -\sum_{p \leq r < q} \mu(p,r) & \text{if } p < q, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition (D., Hohlweg, Pilaud [2016])

The Möbius function of the facial weak order is given by

$$\mu(eW_\varnothing, yW_J) = \begin{cases} (-1)^{|J|}, & \textit{if } y = e, \\ 0, & \textit{otherwise.} \end{cases}$$

Quotients of the facial weak order

Lattice Congruences

Definition

A *lattice congruence* is an equivalence relation $\equiv$ on a lattice $(L, \leq)$ such that for each $x_1 \equiv x_2$ and $y_1 \equiv y_2$ then

1. $x_1 \wedge y_1 \equiv x_2 \wedge y_2$, and
2. $x_1 \vee y_1 \equiv x_2 \vee y_2$.

Theorem (D., Hohlweg, Pilaud [2016])

Given a lattice congruence $\equiv$ on $(W, \leq_R)$, the equivalence classes on $(\mathcal{P}_W, \leq_F)$ defined by

$$xW_I \equiv yW_J \Leftrightarrow x \equiv y \text{ and } xw_{\circ,I} \equiv yw_{\circ,J}$$

give us a lattice congruence.

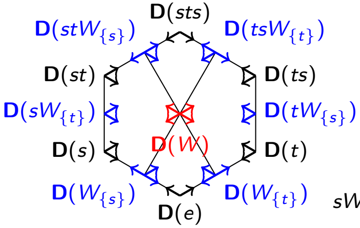
Facial Boolean Lattice

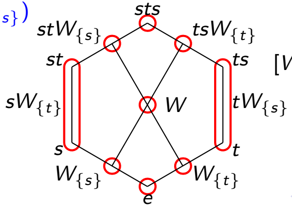
Corollary (D., Hohlweg, Pilaud [2016])

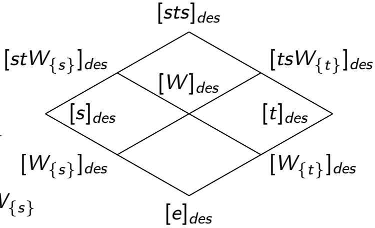
Let the (left) root descent set of a coset xW_I be the set of roots

$$\mathbf{D}(xW_I) := \mathbf{R}(xW_I) \cap \pm\Delta \ \subseteq \ \Phi.$$

Let $xW_I \equiv^{\mathrm{des}} yW_J$ if and only if $\mathbf{D}(xW_I) = \mathbf{D}(yW_J)$.



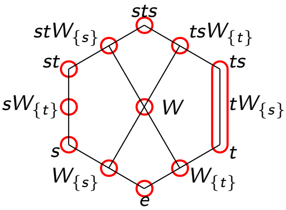


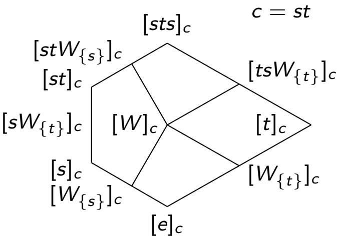


Facial Cambrian Lattice

Corollary (D., Hohlweg, Pilaud [2016])

Let c be any Coxeter element of W. Let $\equiv^c$ be the c-Cambrian congruence (due to Reading [Cambrian Lattice, 2004]). Then let $xW_I \equiv^c yW_J \Leftrightarrow x \equiv^c y$ and $xw_{\circ,I} \equiv^c yw_{\circ,J}$.





Join-Irreducibles

A *join-irreducible* element γ in a poset $(P, \leq)$ is an element with a unique descent $\gamma_\star$.

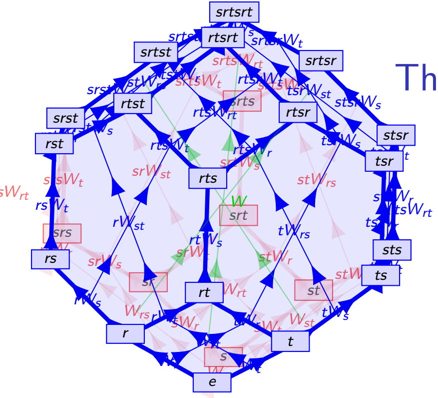
Proposition (D., Hohlweg, Pilaud [2016])

A standard parabolic coxet xW_I is join-irreducible in the facial weak order if and only if we have one of the two following cases

- $I = \varnothing$ *and* x *is join-irreducible in the right weak order, or*
- $I = \{s\}$ *and* xs *is join-irreducible in the right weak order.*

Further Works

- Already extended to hyperplane arrangements and oriented matroids.
- Can we extend the facial weak order to other objects such as arbitrary polytopes?
- Is the facial weak order congruence uniform?



Thank you!