Conjugacy class growth in affine Coxeter groups

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05 September 2023

Coxeter System

- *Coxeter System* (W, S) such that

$$W := \langle s \in S \mid (s_i s_j)^{m_{i,j}} = e \text{ for } s_i, s_j \in S \rangle$$

where $m_{i,j} \in \mathbb{N} \cup \{\infty\}$ and $m_{i,j} = 1$ if and only if $i = j$.

- W – *Coxeter Group*
- S – Set of *simple reflections*

Coxeter System - Two examples

Example (A_2)

Type A_2 has set of simple reflections $S = \{s, t\}$ such that

$$W := \left\langle S \mid s^2 = t^2 = (st)^3 = e \right\rangle$$

Example ($\tilde{A}_2$)

Type $\tilde{A}_2$ has set of simple reflections $S = \{r, s, t\}$ such that

$$W := \left\langle S \mid s^2 = t^2 = (st)^3 = (rt)^3 = (rs)^3 = r^2 = e \right\rangle$$

Two Length Functions

Let (W, S) be a Coxeter system.
Let $w \in W$

- $w = s_1 \dots s_n$ for $s_i \in S$. w has *length* $\ell_S(w) = n$, if n is minimal.

$R = \cup_{w \in W} wSw^{-1}$ is the set of *all reflections*.

- $w = r_1 \dots r_n$ for $r_i \in R$. w has *reflection length* $\ell_R(w) = n$, if n is minimal.

Examples Continued

Example (A_2)

For type A_2 we have $S = \{s, t\}$ and $R = \{s, t, sts\}$. We have the following length functions:

w	$\ell_S(w)$	$\ell_R(w)$
e	0	0
s	1	1
t	1	1
st	2	2
ts	2	2
sts	3	1

Examples Continued

Example ($\tilde{A}_2$)

For type $\tilde{A}_2$ we have $S = \{r, s, t\}$ and $R = \{r, s, t, rsr, rtr, srs, srtrs, rstsr, tsrst, \ldots\}$. Some arbitrary examples:

w	$\ell_S(w)$	$\ell_R(w)$
s	1	1
rs	2	2
rst	3	3
srs	3	1
$rsts$	4	2
$stsrst$	6	4
$strstsrts$	9	1

Growth Functions

How fast do our groups grow with respect to ℓ_S?

Example (D_∞)

$$W = \left\langle \{s, t\} \mid s^2 = t^2 = e \right\rangle$$
$$= \{e, s, t, st, ts, sts, tst, stst, tsts, \ldots\}$$

- 1 element of length 0 (identity)
- 2 elements of length 1 (s, t)
- 2 elements of length 2 (st, ts)
- 2 elements of length 3 (sts, tst)
- etc.

Growth Function

$$B_X(n) = |\{w \in X \mid \ell_S(w) \leq n\}|$$

Example (D_∞)

$$W = \left\langle \{s, t\} \mid s^2 = t^2 = e \right\rangle$$
$$= \{e, s, t, st, ts, sts, tst, stst, tsts, \ldots\}$$

- $B_W(0) = 1$
- $B_W(1) = 3$
- $B_W(2) = 5$
- $B_W(3) = 7$
- $B_W(4) = 9$
- etc.

Growth Rate

How fast does $B_X(n)$ grow?

Example (D_∞)

$$W = \left\langle \{s, t\} \;\middle|\; s^2 = t^2 = e \right\rangle$$
$$= \{e, s, t, st, ts, sts, tst, stst, tsts, \ldots\}$$

- 1, 3, 5, 7, . . . – linear growth – n

Growth Rate

Example ($\tilde{A}_2$)

$$W = \{e, r, s, t, rs, rt, sr, st, ts, tr, srs, tst, srt, \ldots\}$$

- 1 element of length 0
- 3 elements of length 1
- 6 elements of length 2
- 9 elements of length 3
- 12 elements of length 4
- 15 elements of length 5, etc.
- Growth rate: 1, 4, 10, 19, 31, 46, ... – quadratic – n^2

Conjugacy Class

W a Coxeter group

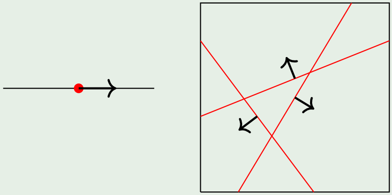
$$C(w) = \left\{ vwv^{-1} \mid v \in W \right\}$$

What is the growth rate of an arbitrary conjugacy class?

Hyperplane Arrangements

- E Euclidean space with underlying Euclidean vector space $(V, \langle\cdot,\cdot\rangle)$.
- A *hyperplane H* is a $\operatorname{codim} 1$ subspace of V.
- A *(hyperplane) arrangement* is a *finite* collection of hyperplanes.

Example

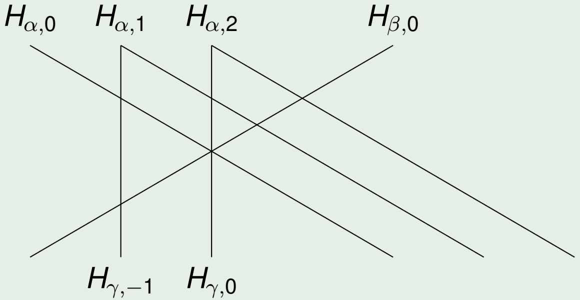


Hyperplanes and vectors

For $\alpha \in V$ a vector.

- $H_{\alpha,k} = \{\lambda \in V \mid \langle \alpha, \lambda \rangle = k\}$ - hyperplane.
- $H_\alpha = H_{\alpha,0}$ – central hyperplane.
- s_α – reflection fixing H_α pointwise.

Example



Root Systems

Definition

A *root system* Φ is (finite) collection of nonzero vectors satisfying:

1. $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for every $\alpha \in \Phi$.
2. $s_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$.
3. $\frac{2\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle} \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.

The $\alpha \in \Phi$ are called *roots*.

- Φ^+ – Positive roots
- Φ^- – Negative roots
- Δ – Simple roots
- $W = \langle S \rangle$, $S = \{s_\alpha \mid \alpha \in \Delta\}$ – (finite) Coxeter group.
- $R = \{s_\alpha \mid \alpha \in \Phi^+\}$

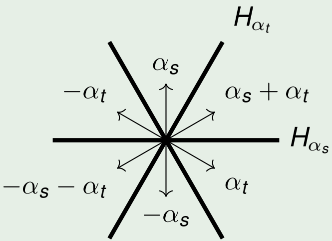
Coxeter Arrangements

Definition

A *Coxeter arrangement* is the arrangement for a root system Φ:

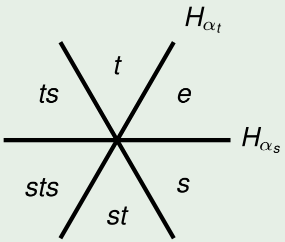
$$\mathcal{A}(\Phi) = \{H_\alpha \mid \alpha \in \Phi^+\}.$$

Example (A_2 Coxeter Arrangement)



A_2 Coxeter Arrangement

Example (A_2 Coxeter Arrangement)



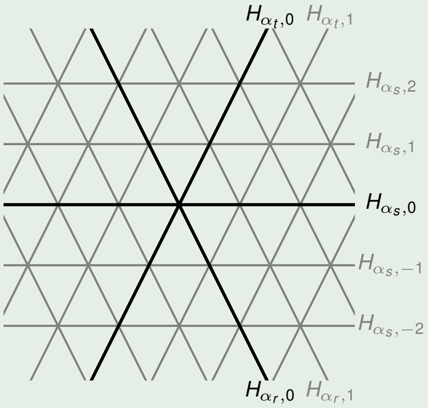
Affine Coxeter Group

Let Φ be root system of finite Coxeter group.

- For $\alpha \in \Phi^+$ let $H_{\alpha,j} = \{\lambda \in V \mid \langle \lambda, \alpha \rangle = j\}$ be affine hyperplane in V.
- $r_{\alpha,j}$ reflection fixing $H_{\alpha,j}$ pointwise.
- R is collection of $r_{\alpha,j}$ for $\alpha \in \Phi^+$, $j \in \mathbb{Z}$.
- *Affine Coxeter group* $W = \langle R \rangle$.

Affine Coxeter Arrangement

Example ($\tilde{A}_2$ Arrangement)



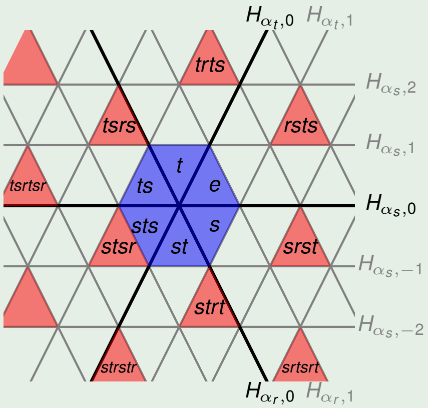
Finite part + Translations

W an affine Coxeter group.

- W_0 – finite part generated by $R_0 = \{r_{\alpha,0} \mid \alpha \in \Phi^+\}$.
- $\pi : W \to W_0$ a projection sending $r_{\alpha,j} \mapsto r_{\alpha,0}$.
- Kernal of π are *translations T* – $W_0 \cong W/T$.
- T is a free abelian normal subgroup.

Example Continued

Example

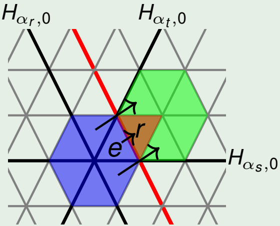


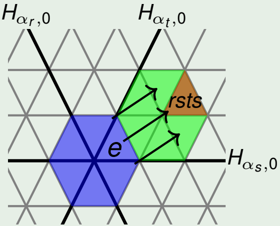
Move Set + Fixed Space

For W an affine Coxeter group and $w \in W$

$$\mathrm{Mov}(w) = \{\lambda \in V \mid w(x) = x + \lambda \text{ for some } x \in E\}$$
$$\mathrm{Fix}(w) = \{x \in E \mid w(x) = x\}$$

Example





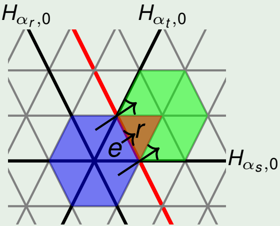
$$\mathrm{Mov}(r) = \{a\lambda \mid a \in \mathbb{R}\} \qquad \mathrm{Mov}(rsts) = \{\lambda\}$$
$$\mathrm{Fix}(r) = H_{\alpha_r,1} \qquad \mathrm{Fix}(rsts) = \varnothing$$

Elliptic + Translation

For W an affine Coxeter group and $w \in W$

- *Elliptic* – $\mathrm{Fix}(w) \neq \varnothing$
- *Translation* – $|\mathrm{Mov}(w)| \leq 1$

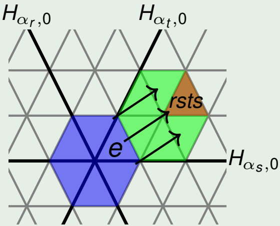
Example



$\mathrm{Mov}(r) = \{a\lambda \mid a \in \mathbb{R}\}$

$\mathrm{Fix}(r) = H_{\alpha_r,1}$

Elliptic



$\mathrm{Mov}(rsts) = \{\lambda\}$

$\mathrm{Fix}(rsts) = \varnothing$

Translation

Factorisation

W an affine Coxeter group.
Recall:

- W_0 – finite part
- T – Translations
- $W_0 \cong W/T$

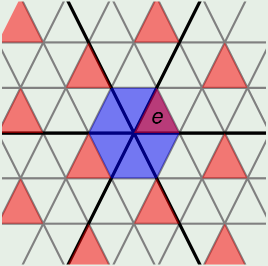
Factorisation

W an affine Coxeter group.

- W_0 – finite part – Elliptic
- T – Translations
- $W_0 \cong W/T$ – semidirect product $W = T \rtimes W_0$

For $w \in W$ we have $w = tu$ with $t \in T$, $u \in W_0$.

Example



Main Theorem

Recall:

- Conjugacy class: $C(w) = \left\{ vwv^{-1} \mid v \in W \right\}$
- Growth rate: How fast does $B_W(n) = |\{w \in W \mid \ell_S(w) \leq n\}|$ grow?
- Reflection length: Length in generating set $R = \cup_{w \in W} wSw^{-1}$.

Theorem (D., Evetts '23)

Let $W = T \rtimes W_0$ be an affine Coxeter group with translations T and finite part W_0. Let $w = tu \in W$ with $t \in T$ and $u \in W_0$.

$$\text{Conjugacy class growth (over } S) = n^{\ell_R(u)}$$

where $\ell_R(u)$ is reflection length.

Examples

Example (Elliptic)

- $w = st = (e)(st)$
- $C(w) = \{st, ts, rtsr, rstr, rtrsrt, srtrsr, trsrtr, rsrtrs, \dots\}$
- Num elts: 2, 2, 4, 6, 6, 8, 10, 10, ...
- $B_{C(w)}(n) \to 2, 4, 8, 14, 20, 28, 38, 48, \dots$
- Quadratic growth – $\ell_R(w) = 2$.

Examples

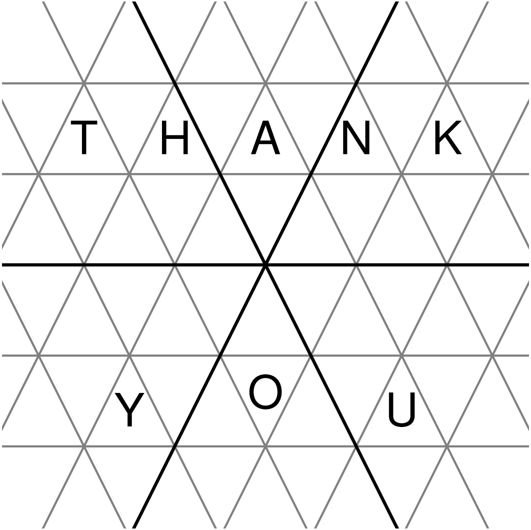
Example (Translation)

- $w = rsts = (rsts)(e)$
- $C(w) = \{rsts, rsrt, stsr, srtr, trsr, rtrs\}$
- Num elts: 6
- $B_{C(w)}(n) \to 6, 6, 6, 6, \ldots$
- Constant growth – $\ell_R(e) = 0$.

Examples

Example

- $w = rst = (rsts)(s)$
- $C(w) = \{rst, str, trs, tsrts, srtsr, tsrtr, rsrts, rtsrt, rtrsr, \dots\}$
- Num elts: 3, 6, 3, 3, 3, 3, 3, …
- $B_{C(w)}(n) \to 3, 9, 12, 15, 18, 21, \dots$
- Linear growth – $\ell_R(s) = 1$.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