The importance of being *straight*

Aram Dermenjian

York University

14 April 2021

And so it begins. . .

A basic human problem

A basic human problem

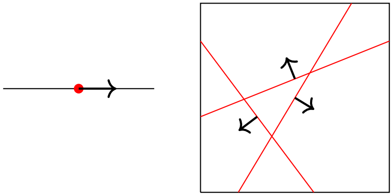
A basic human problem

A basic human problem

What is a hyperplane?

- $(V, \langle \cdot, \cdot \rangle)$ - n-dim real Euclidean vector space.
- A *hyperplane H* is $\dim\ n-1$ subspace of V with normal e_H.

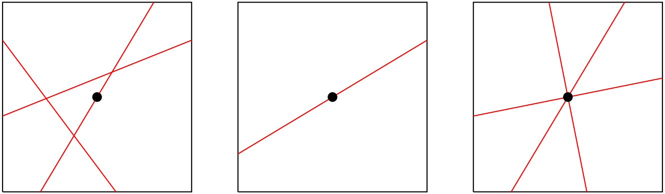
Example



Arranging hyperplanes

- A *hyperplane arrangement* is $\mathcal{A} = \{H_1, H_2, \ldots, H_k\}$.
- $\mathcal{A}$ is *central* if $\{0\} \subseteq \bigcap \mathcal{A}$.
- $\mathcal{A}$ is *essential* if normal vectors span V.

Example



Not central
Essential

Central
Not essential

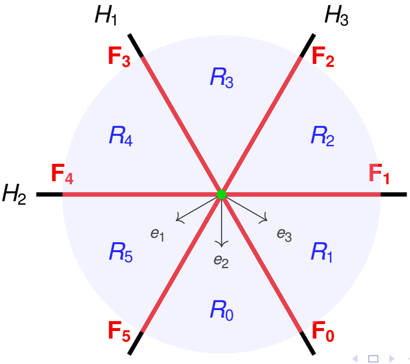
Central
Essential

In terms of food?

Central essential hyperplane arrangement

Exploding arrangements

- *Regions* $\mathscr{R}$ - closures of connected components of V without $\mathcal{A}$.
- *Faces* $\mathscr{F}$ - intersections of some regions.

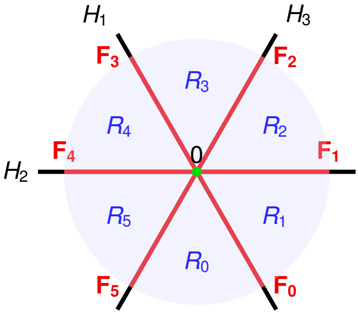


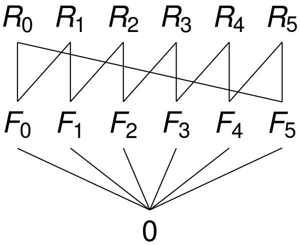
Fraternal Twins

Same Combinatorial Family - Face Poset

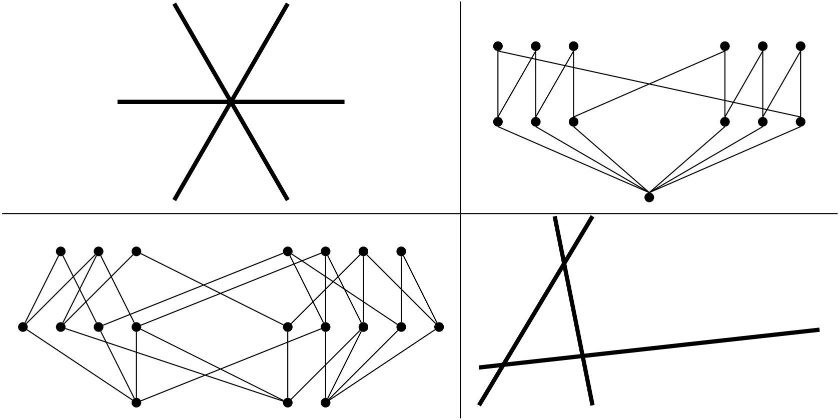
A *poset* $(P, \leq)$ is a set P with a partial order $\leq$:

- $x \leq x$ for all $x \in P$
- $x \leq y$ and $y \leq x$ implies $x = y$ for all $x, y \in P$
- $x \leq y$ and $y \leq z$ implies $x \leq z$ for all $x, y, z \in P$





Different Families



How important is straightness?

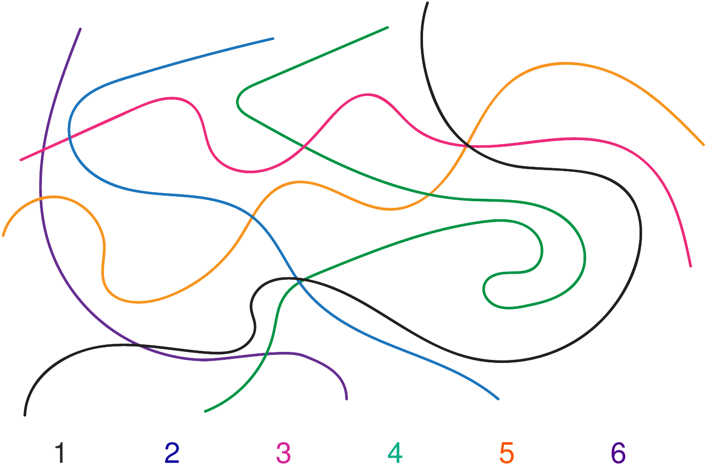
The importance of being straight[1]

Branko Grünbaum
University of Washington

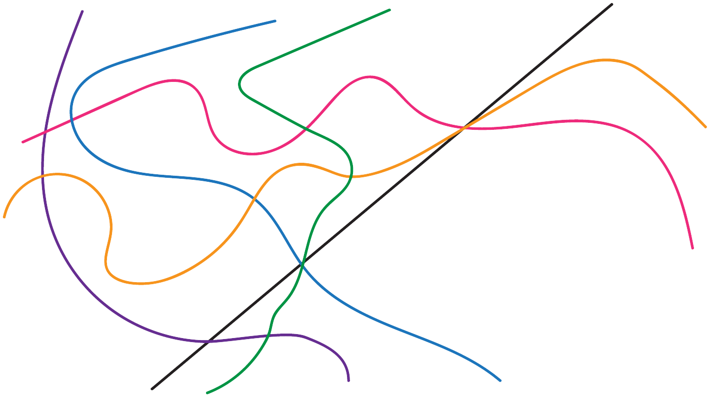
How important is straightness?

[235] BRANKO GRÜNBAUM: *The importance of being straight (?)*, in: "Proc. of the Twelfth Biannual Intern. Seminar of the Canadian Math. Congress," *Time Series and Stochastic Processes; Convexity and Combinatorics* (R. Pyke, ed.), (Vancouver 1969), Canadian Math. Congress, Montreal 1970, pp. 243–254. (280)

Can straightness be forced?

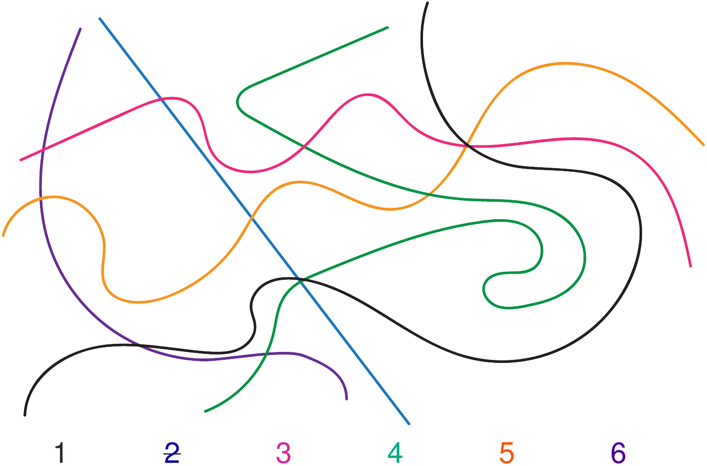


Can straightness be forced?

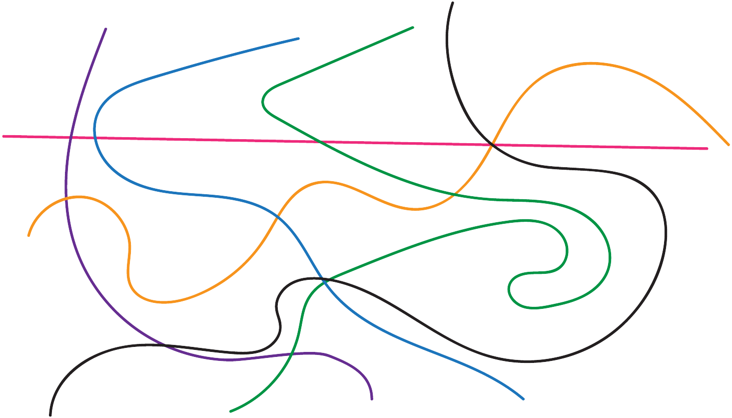


~~1~~ 2 3 4 5 6

Can straightness be forced?

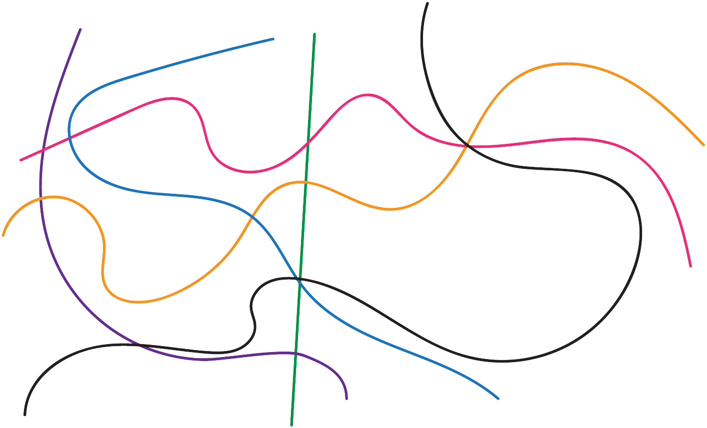


Can straightness be forced?



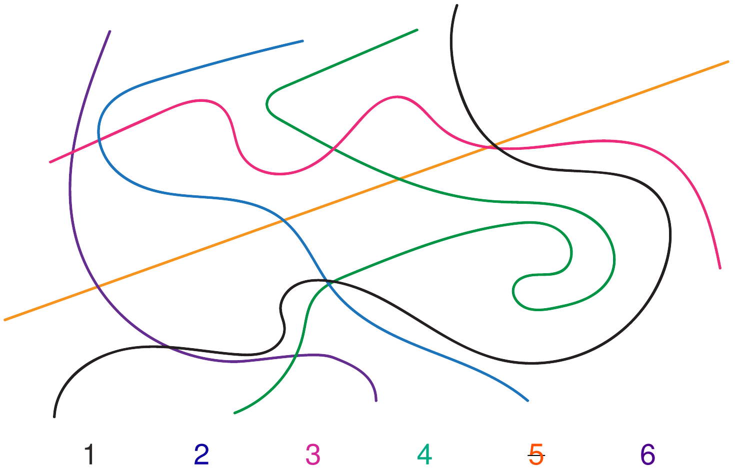
1 2 ~~3~~ 4 5 6

Can straightness be forced?

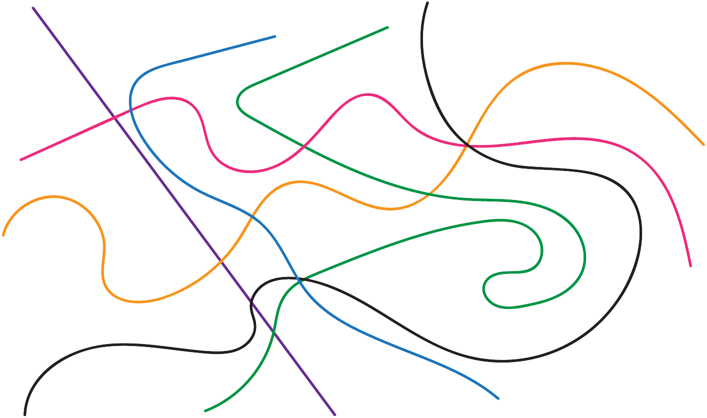


1 2 3 4 5 6

Can straightness be forced?

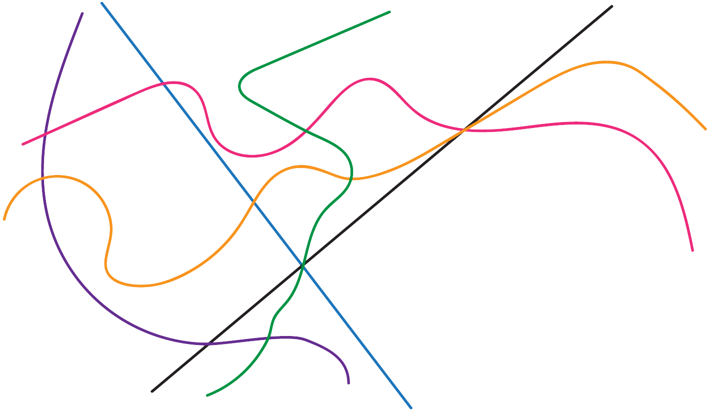


Can straightness be forced?



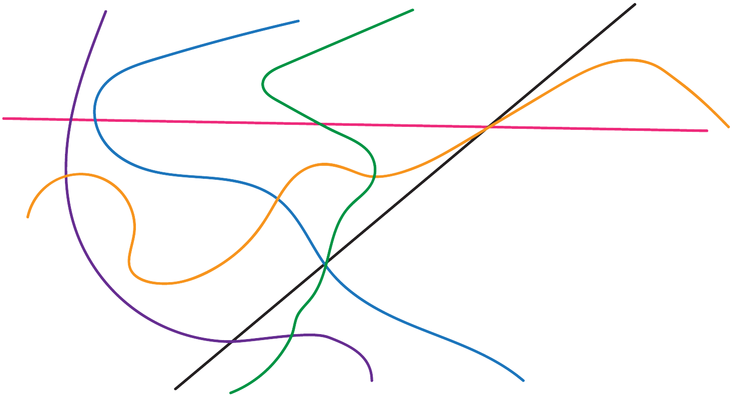
1 2 3 4 5 ~~6~~

Can straightness be forced?



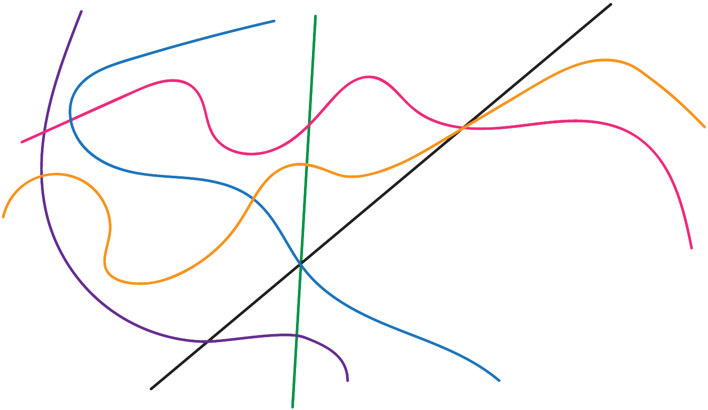
~~1~~ ~~2~~ 3 4 5 6

Can straightness be forced?



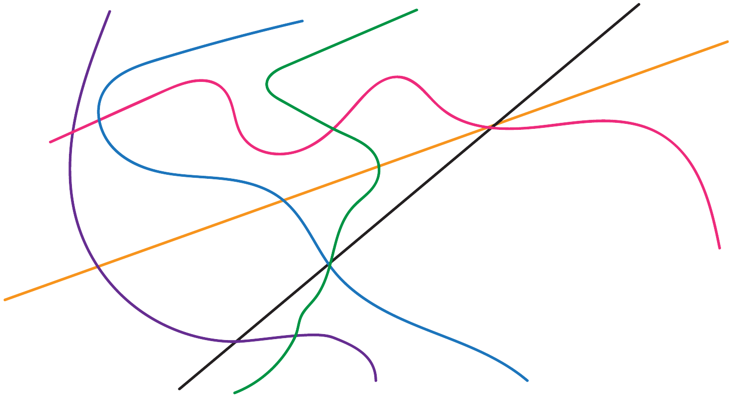
~~1~~ 2 ~~3~~ 4 5 6

Can straightness be forced?



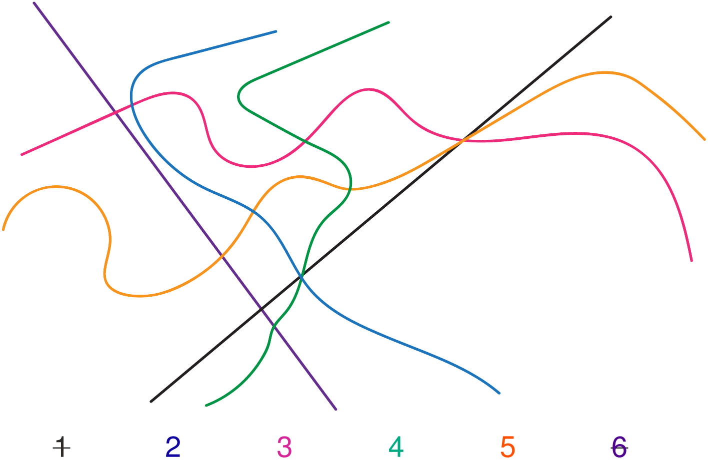
~~1~~ 2 3 ~~4~~ 5 6

Can straightness be forced?

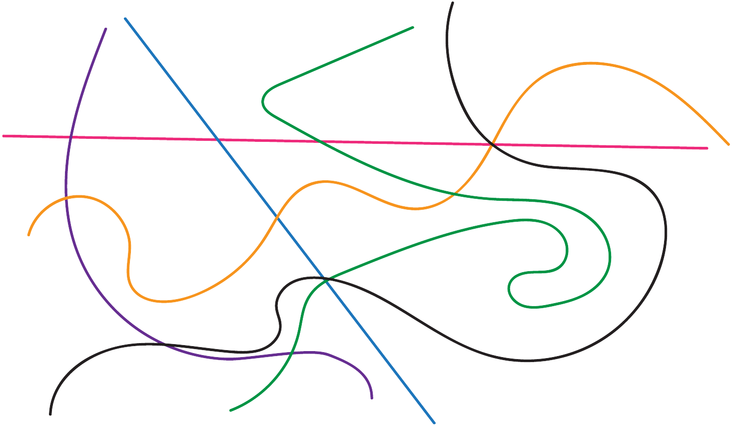


~~1~~ 2 3 4 ~~5~~ 6

Can straightness be forced?

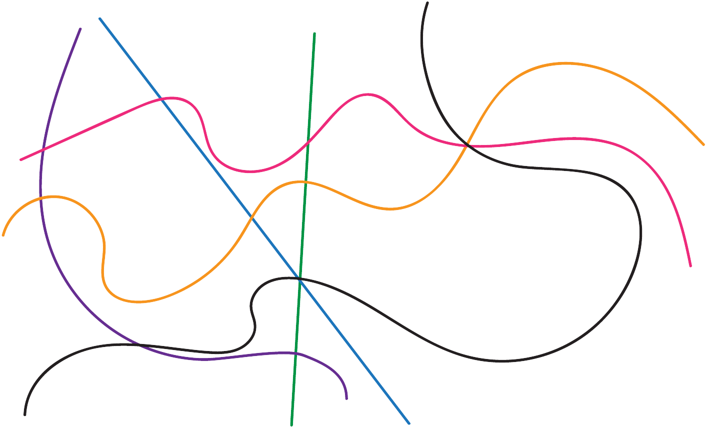


Can straightness be forced?



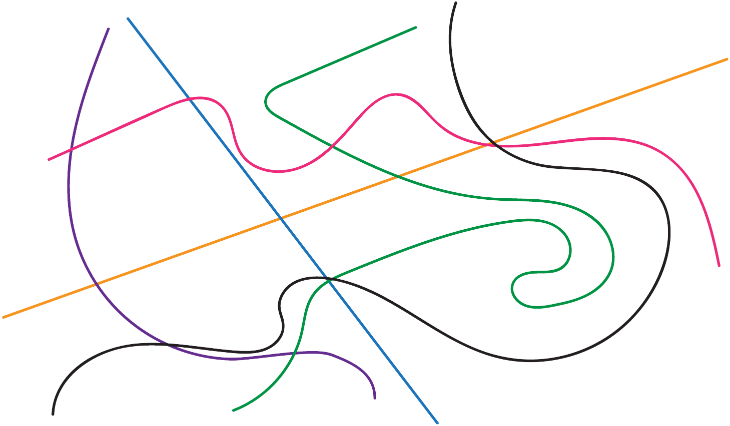
1 ~~2~~ ~~3~~ 4 5 6

Can straightness be forced?



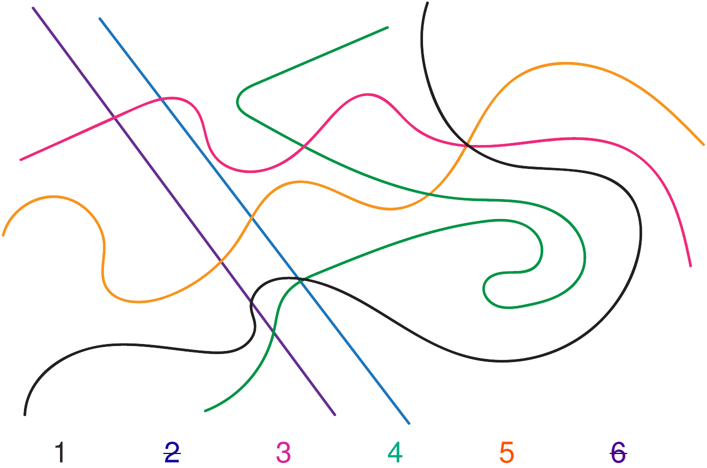
1 ~~2~~ 3 ~~4~~ 5 6

Can straightness be forced?

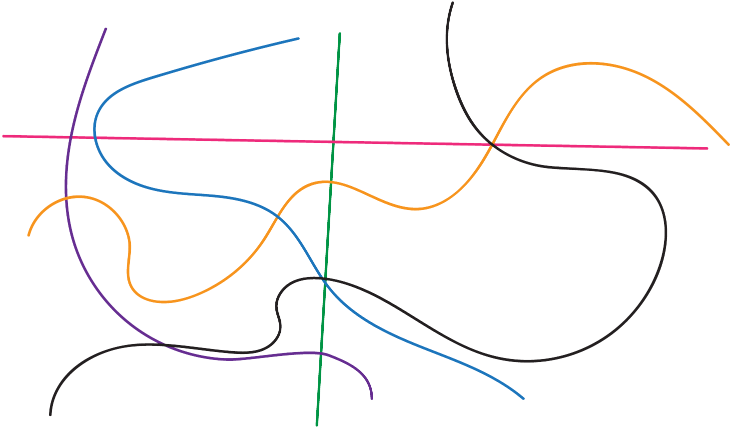


1 ~~2~~ 3 4 ~~5~~ 6

Can straightness be forced?

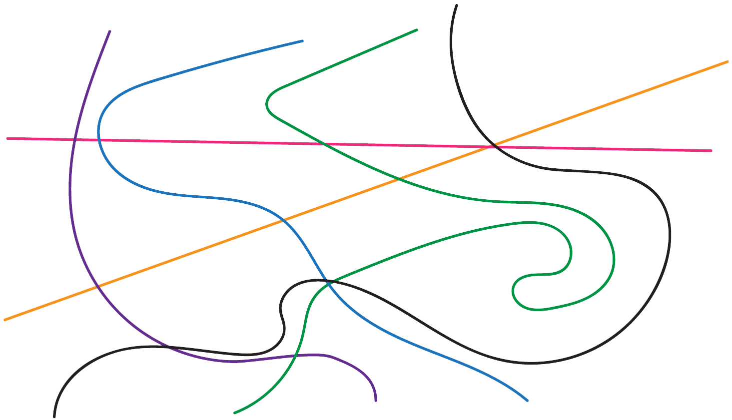


Can straightness be forced?



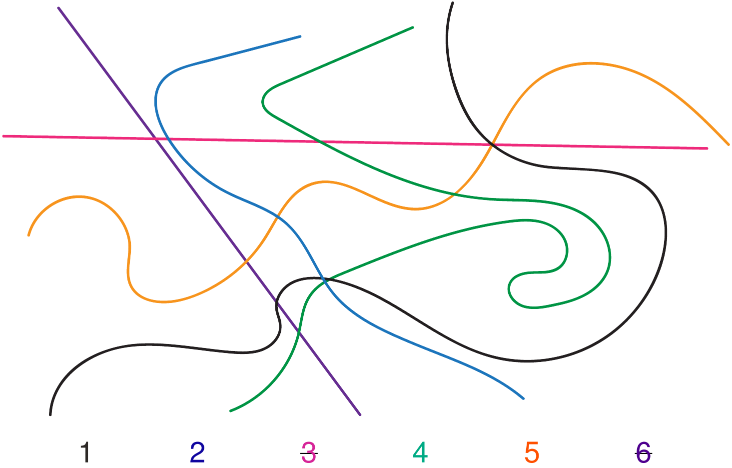
1 2 ~~3~~ ~~4~~ 5 6

Can straightness be forced?

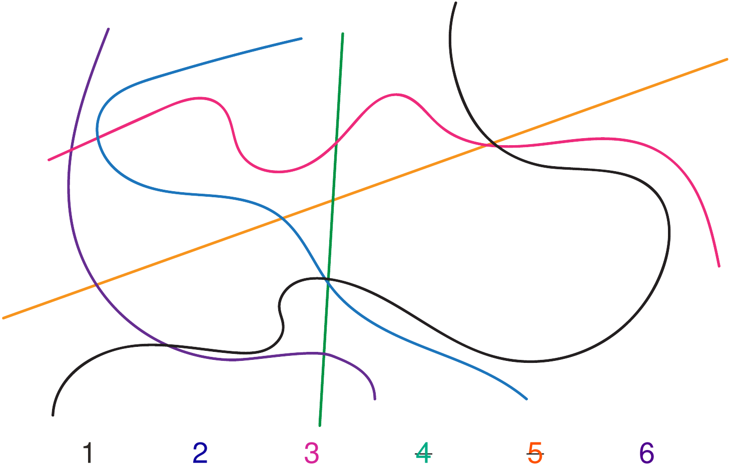


1 2 ~~3~~ 4 ~~5~~ 6

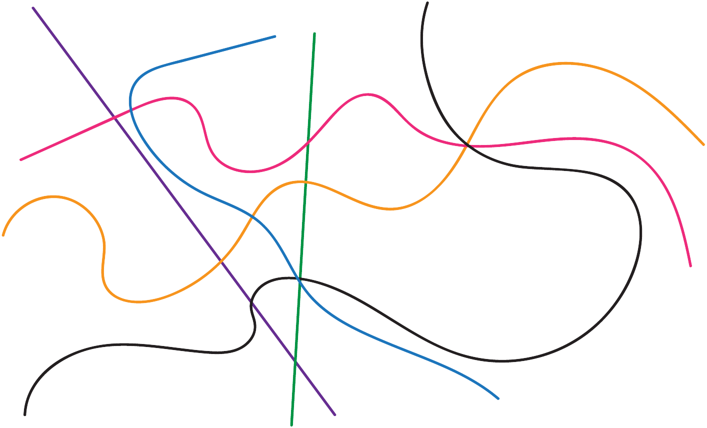
Can straightness be forced?



Can straightness be forced?

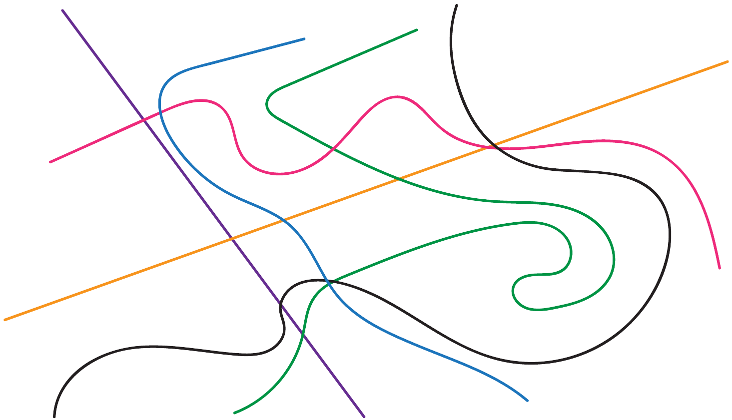


Can straightness be forced?



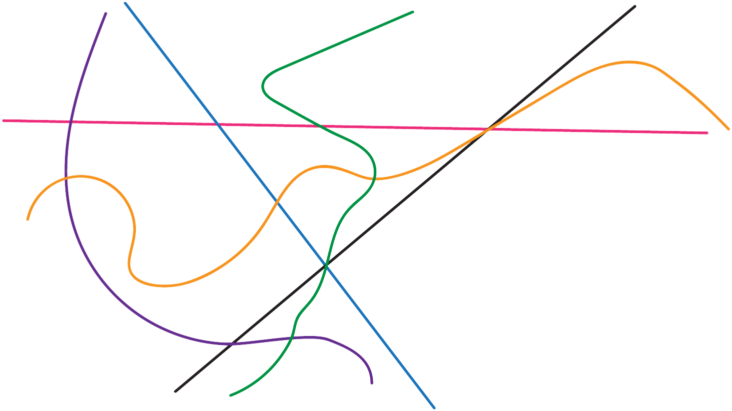
1 2 3 ~~4~~ 5 ~~6~~

Can straightness be forced?



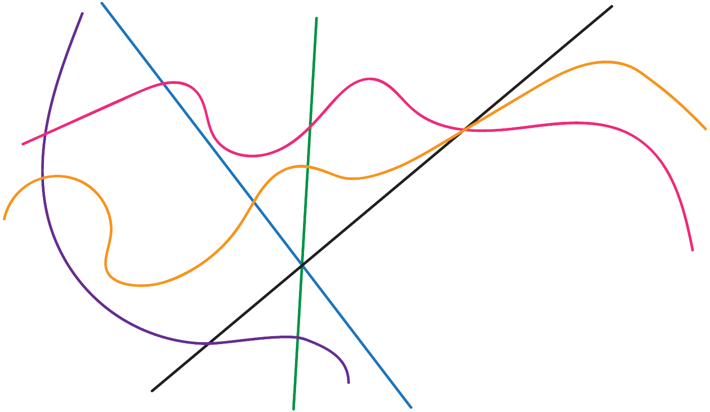
1 2 3 4 ~~5~~ ~~6~~

Can straightness be forced?

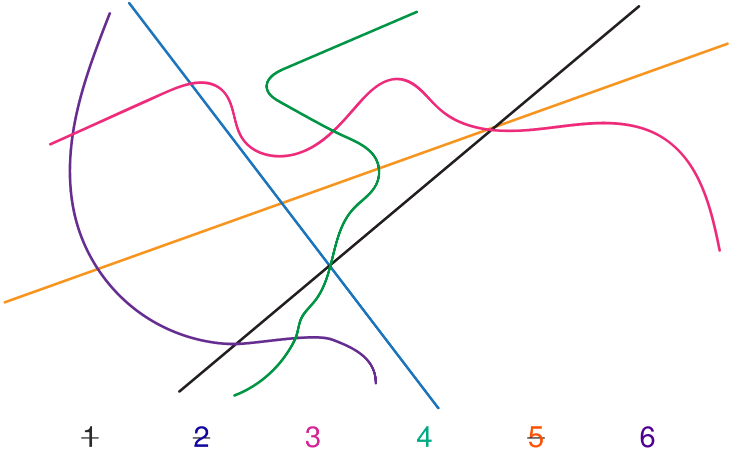


~~1~~ ~~2~~ ~~3~~ 4 5 6

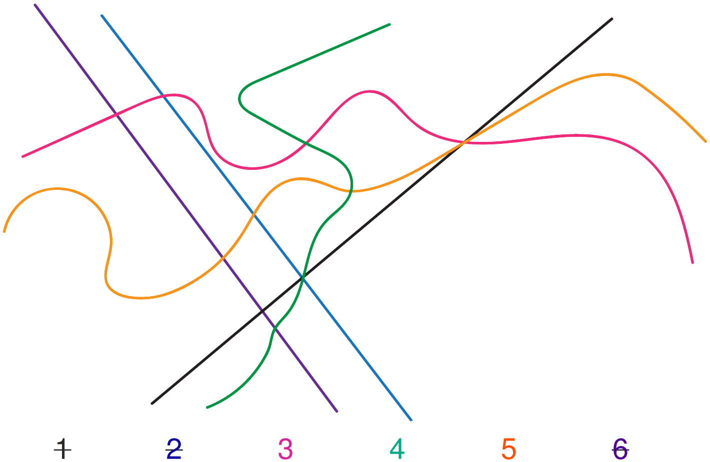
Can straightness be forced?



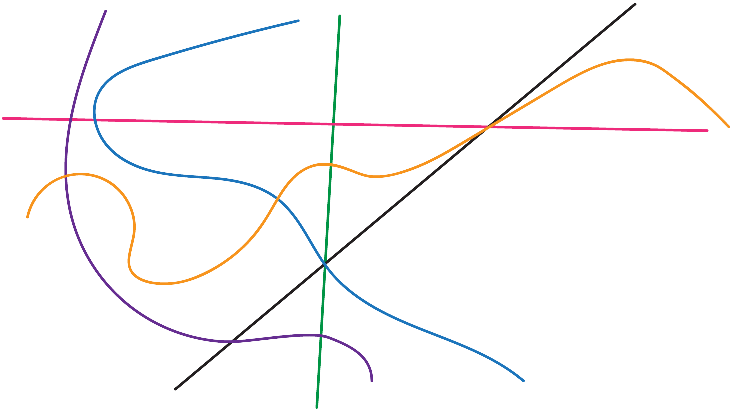
Can straightness be forced?



Can straightness be forced?

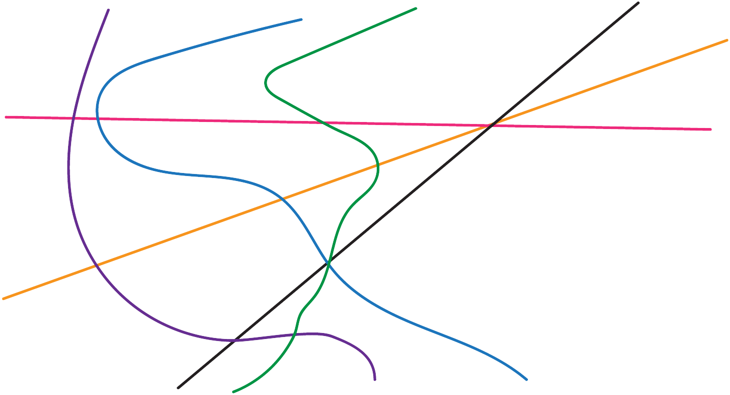


Can straightness be forced?



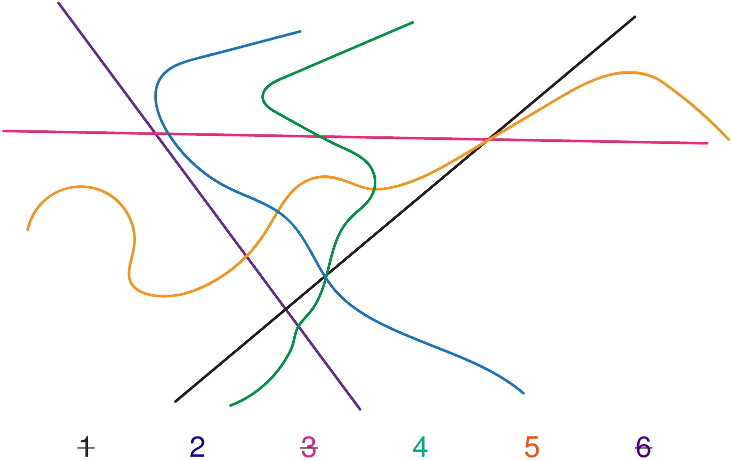
~~1~~ 2 ~~3~~ ~~4~~ 5 6

Can straightness be forced?

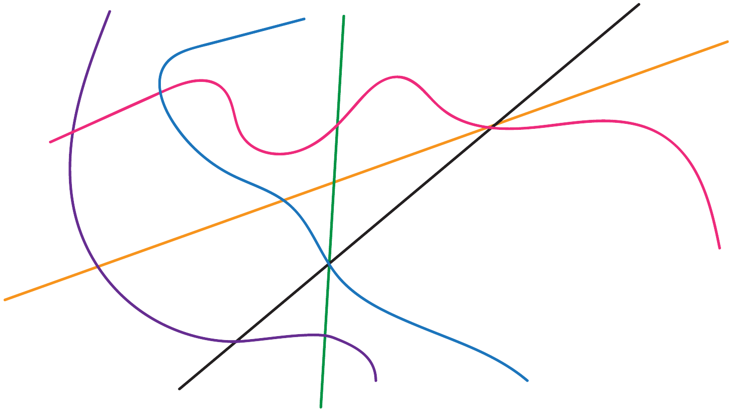


~~1~~ 2 ~~3~~ 4 ~~5~~ 6

Can straightness be forced?

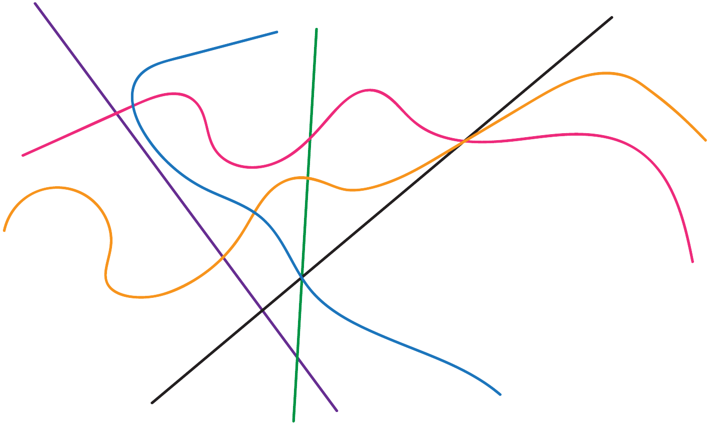


Can straightness be forced?



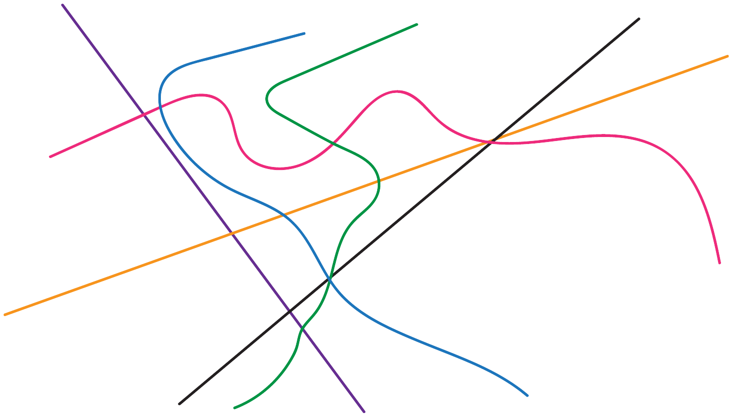
~~1~~ 2 3 ~~4~~ ~~5~~ 6

Can straightness be forced?



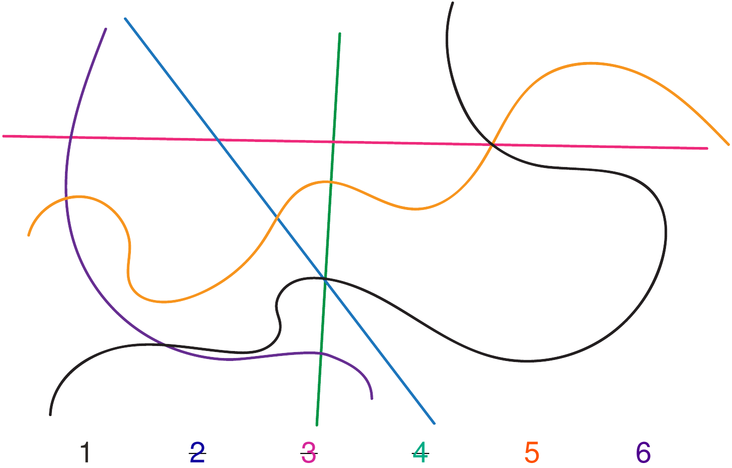
~~1~~ 2 3 ~~4~~ 5 ~~6~~

Can straightness be forced?

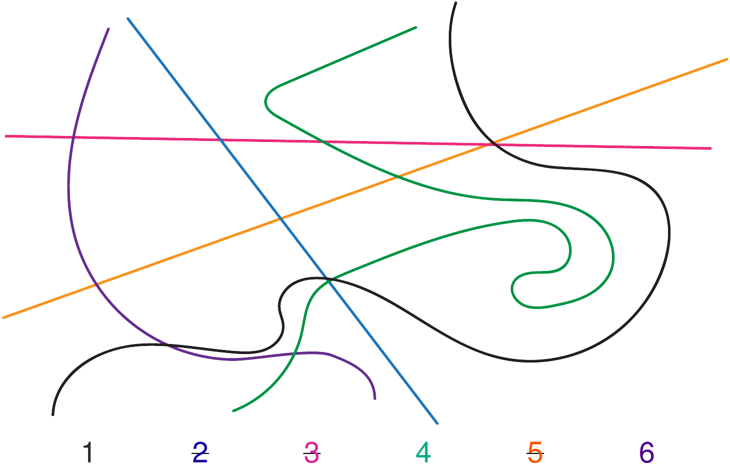


~~1~~ 2 3 4 ~~5~~ ~~6~~

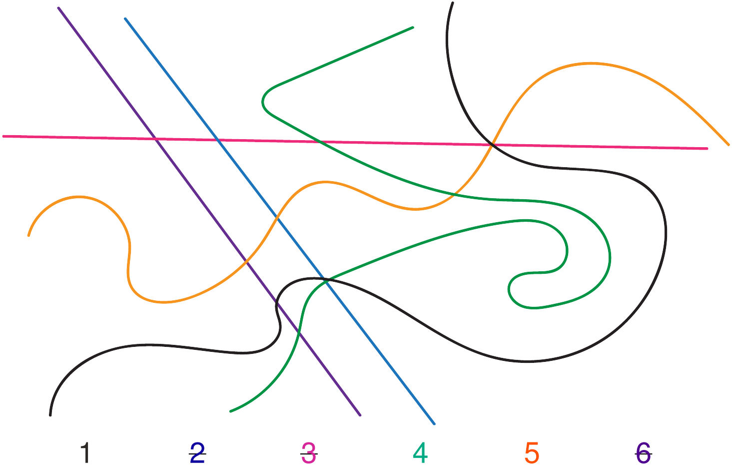
Can straightness be forced?



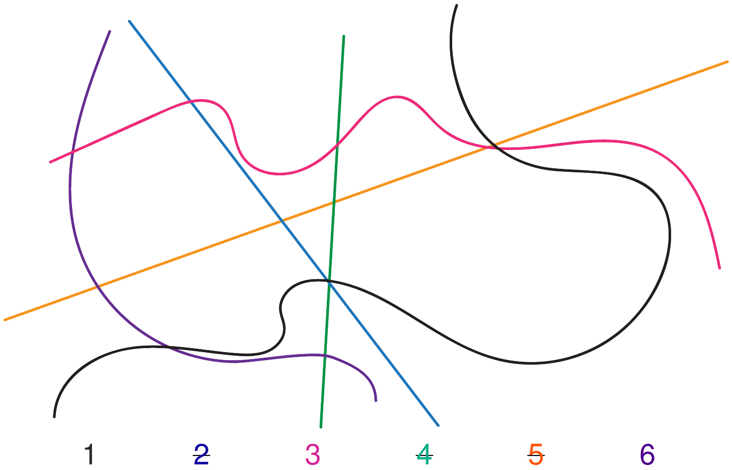
Can straightness be forced?



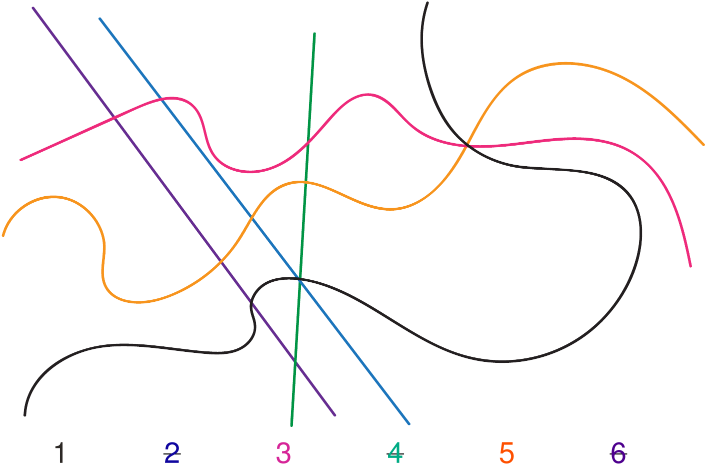
Can straightness be forced?



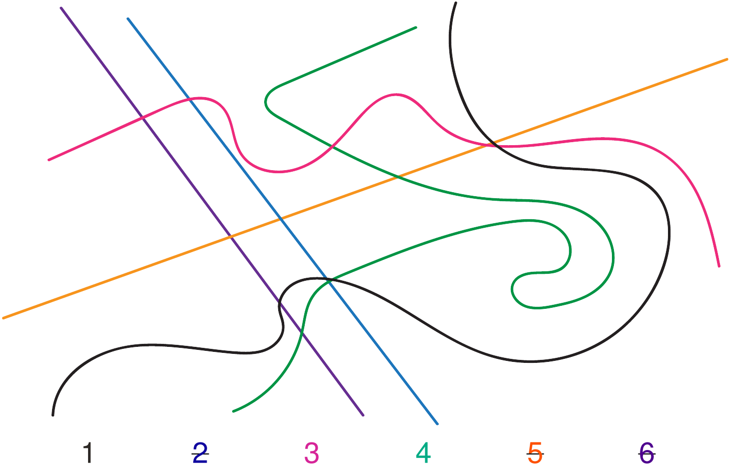
Can straightness be forced?



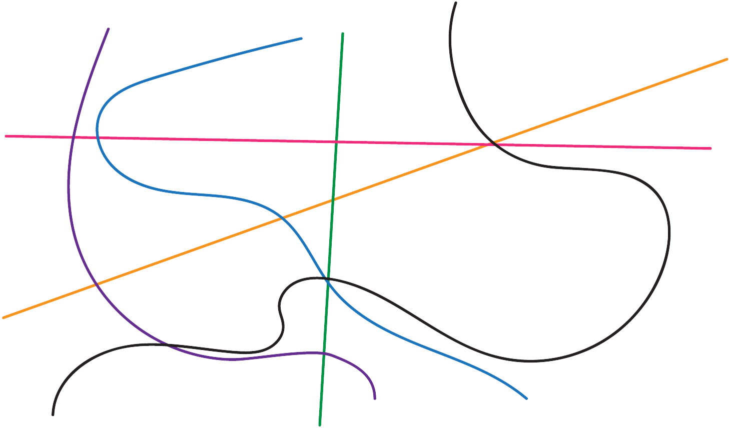
Can straightness be forced?



Can straightness be forced?

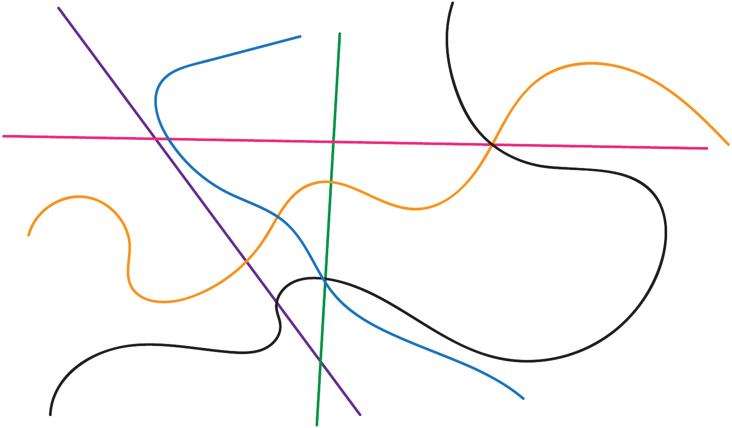


Can straightness be forced?



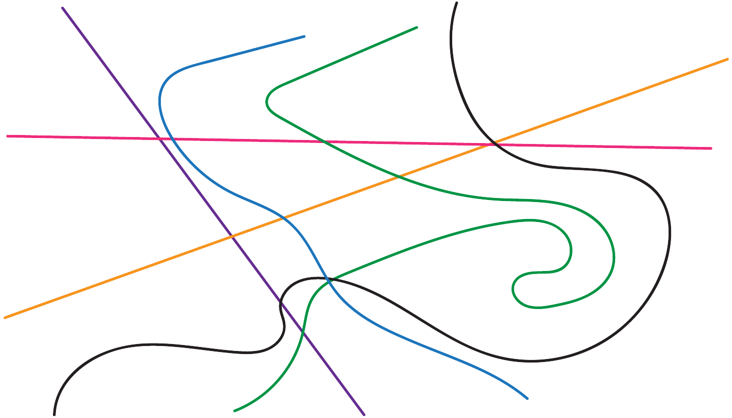
1 2 ~~3~~ ~~4~~ ~~5~~ 6

Can straightness be forced?



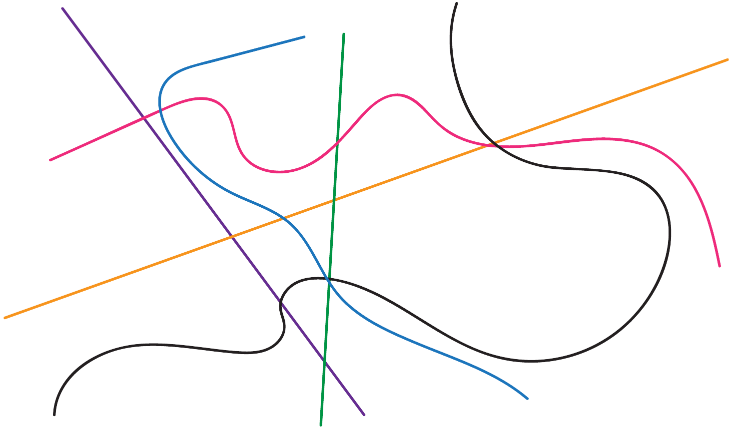
1 2 ~~3~~ ~~4~~ 5 ~~6~~

Can straightness be forced?



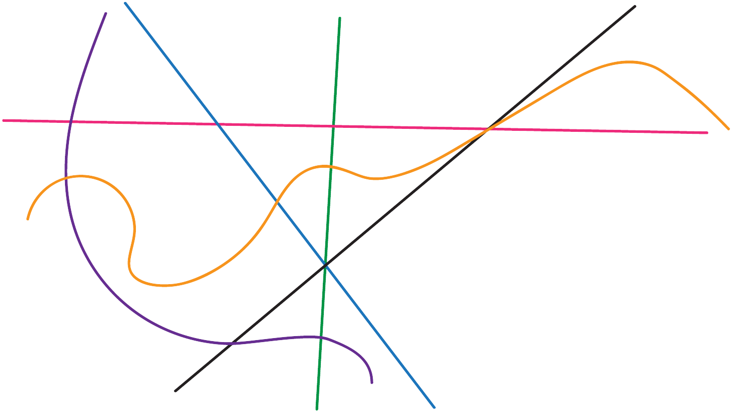
1 2 ~~3~~ 4 ~~5~~ ~~6~~

Can straightness be forced?



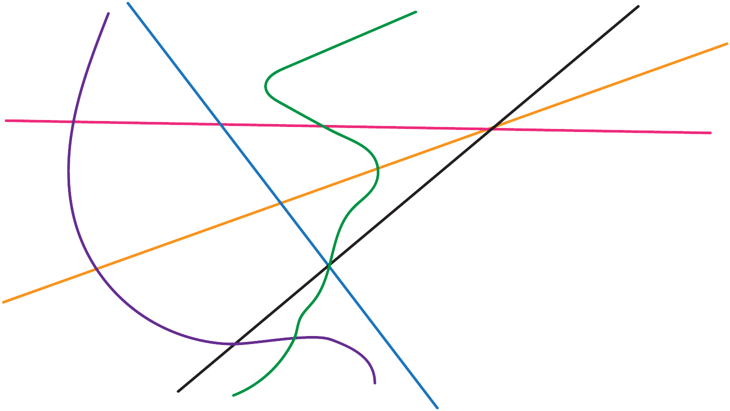
1 2 3 ~~4~~ ~~5~~ ~~6~~

Can straightness be forced?



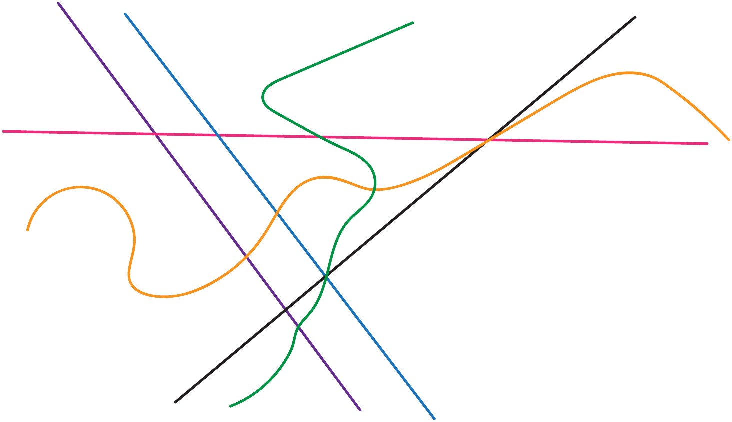
~~1~~ ~~2~~ ~~3~~ ~~4~~ 5 6

Can straightness be forced?



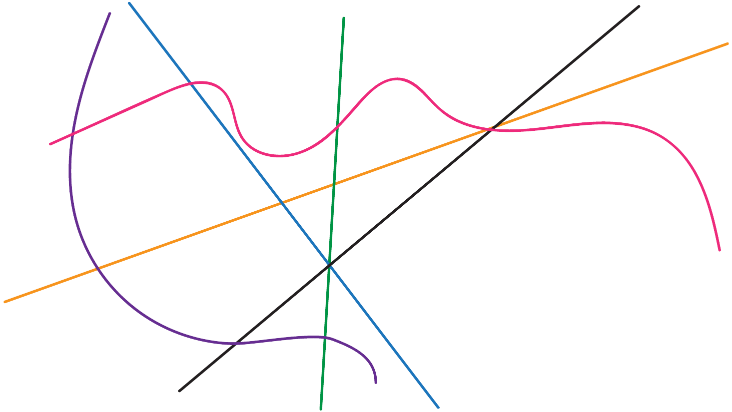
~~1~~ ~~2~~ ~~3~~ 4 ~~5~~ 6

Can straightness be forced?



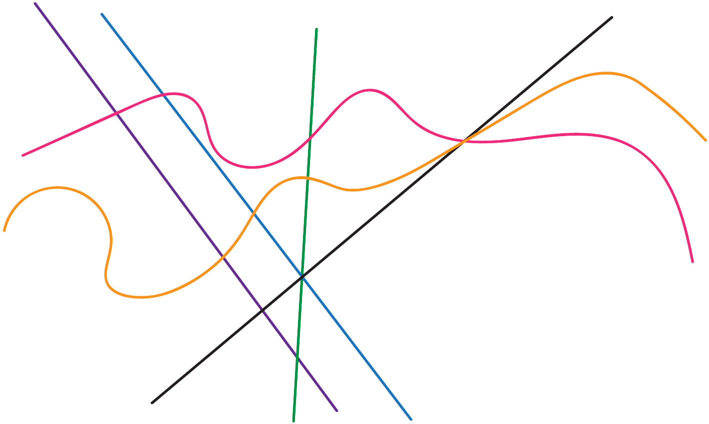
~~1~~ ~~2~~ ~~3~~ 4 5 ~~6~~

Can straightness be forced?



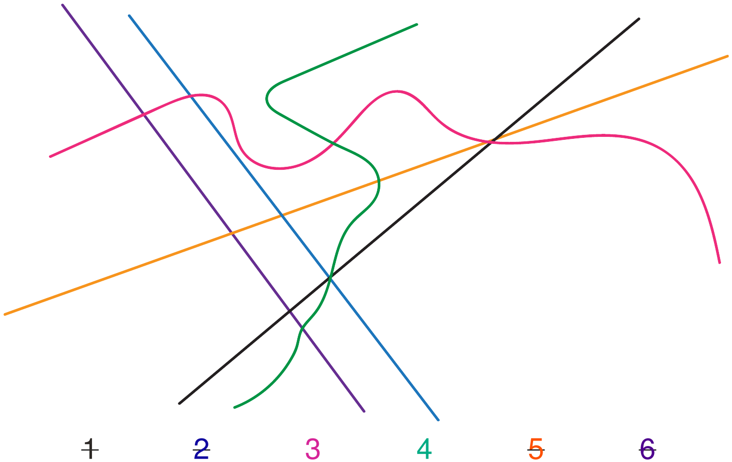
~~1~~ ~~2~~ 3 ~~4~~ ~~5~~ 6

Can straightness be forced?

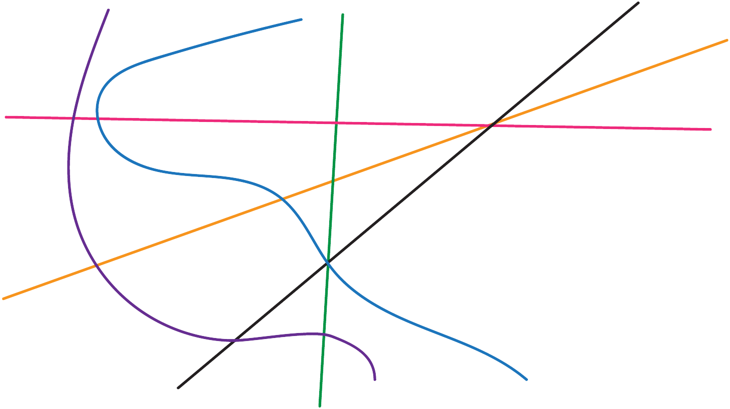


~~1~~ ~~2~~ 3 ~~4~~ 5 ~~6~~

Can straightness be forced?

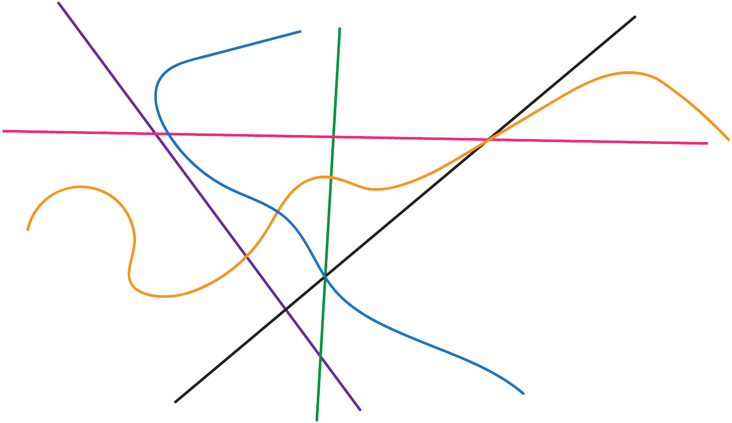


Can straightness be forced?



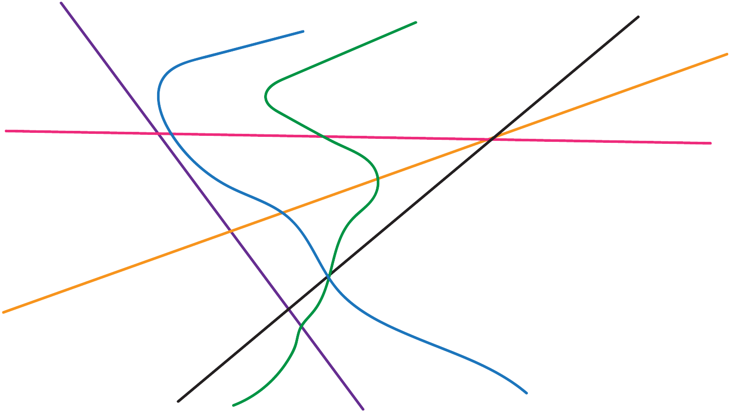
~~1~~ 2 ~~3~~ ~~4~~ ~~5~~ 6

Can straightness be forced?



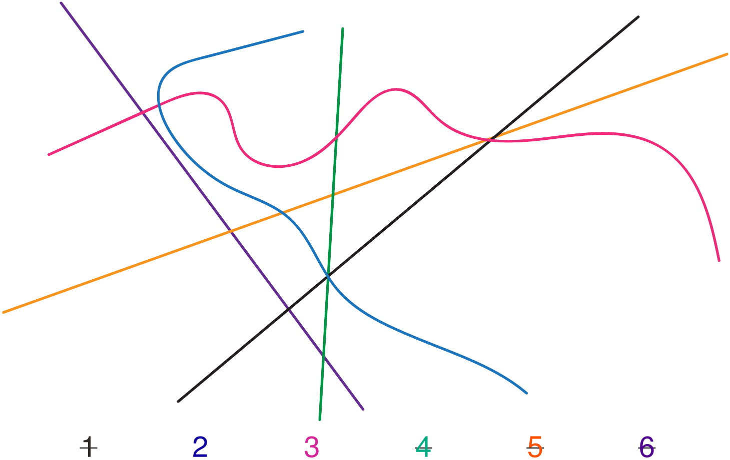
~~1~~ 2 ~~3~~ ~~4~~ 5 ~~6~~

Can straightness be forced?

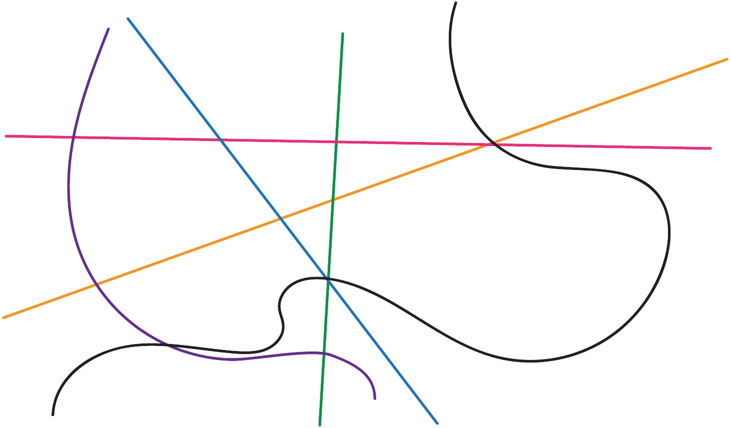


~~1~~ 2 ~~3~~ 4 ~~5~~ ~~6~~

Can straightness be forced?

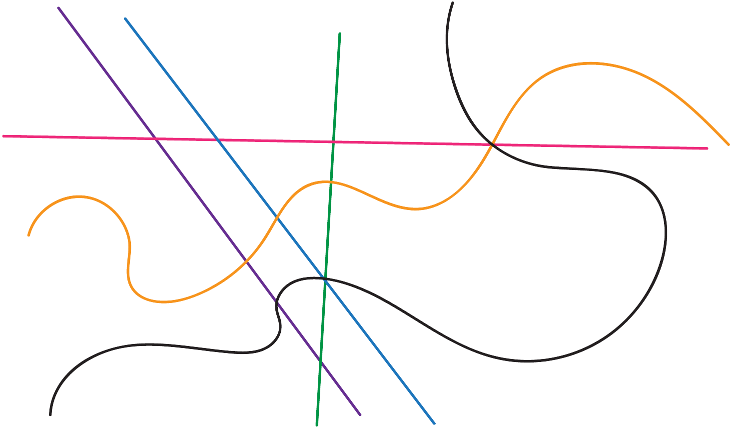


Can straightness be forced?



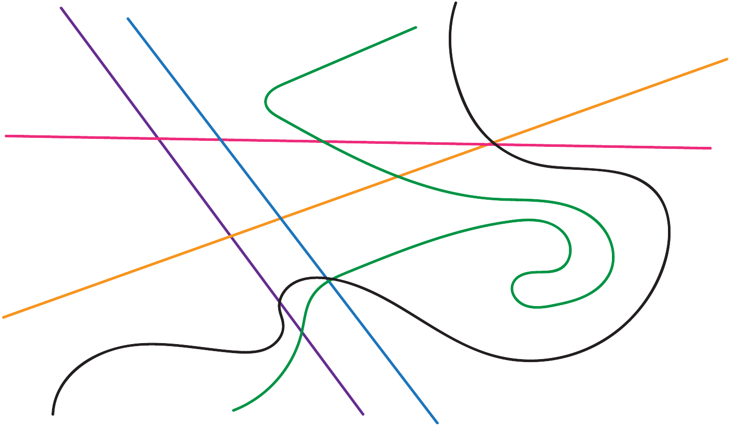
1 ~~2~~ ~~3~~ ~~4~~ ~~5~~ 6

Can straightness be forced?



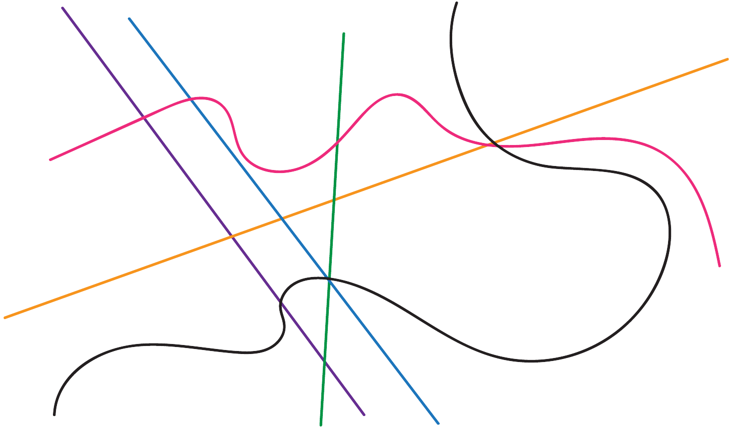
1 ~~2~~ ~~3~~ ~~4~~ 5 ~~6~~

Can straightness be forced?



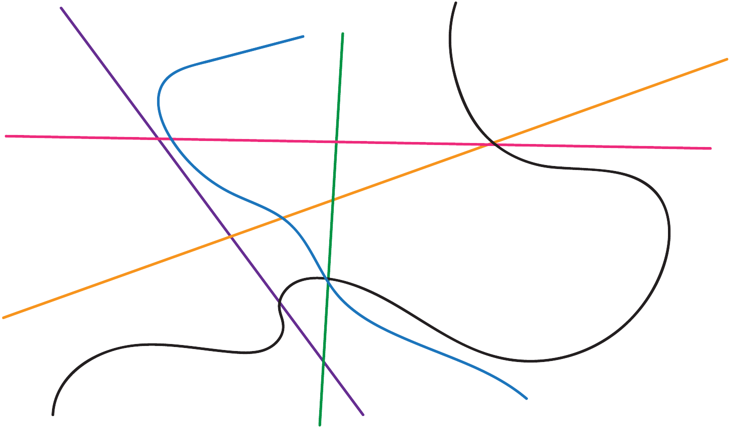
1 ~~2~~ ~~3~~ 4 ~~5~~ ~~6~~

Can straightness be forced?



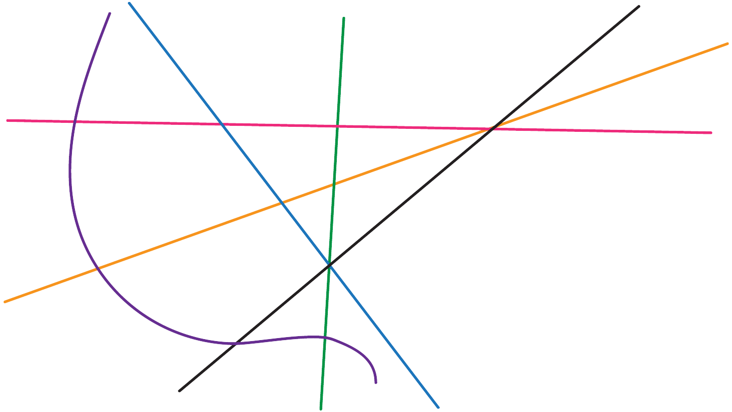
1 ~~2~~ 3 ~~4~~ ~~5~~ ~~6~~

Can straightness be forced?



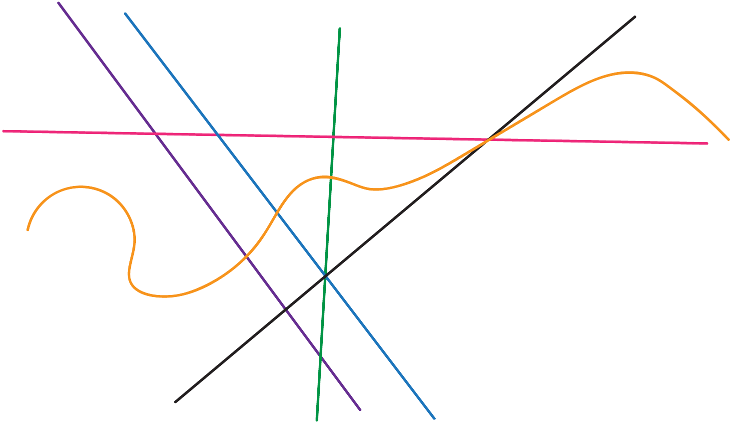
1 2 ~~3~~ ~~4~~ ~~5~~ ~~6~~

Can straightness be forced?



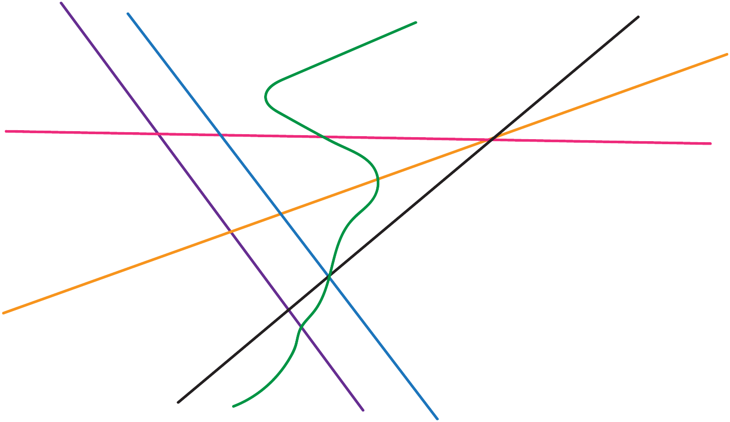
~~1~~ ~~2~~ ~~3~~ ~~4~~ ~~5~~ 6

Can straightness be forced?



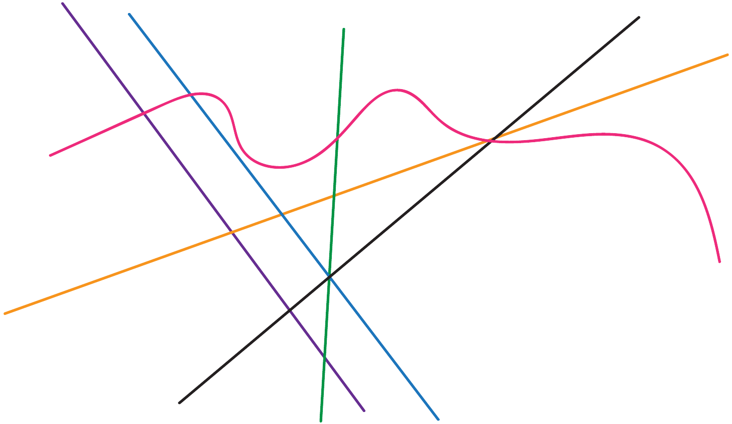
~~1~~ ~~2~~ ~~3~~ ~~4~~ 5 ~~6~~

Can straightness be forced?



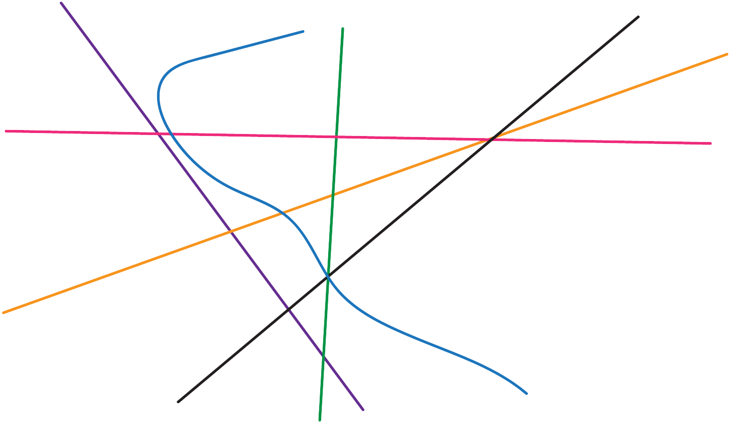
~~1~~ ~~2~~ ~~3~~ 4 ~~5~~ ~~6~~

Can straightness be forced?



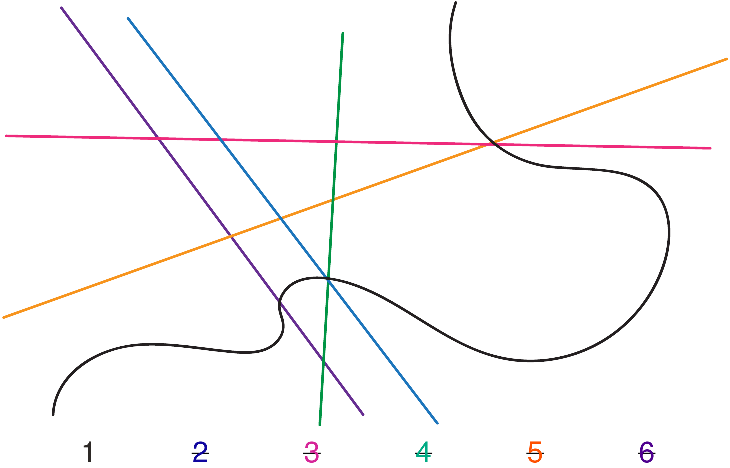
~~1~~ ~~2~~ 3 ~~4~~ ~~5~~ ~~6~~

Can straightness be forced?

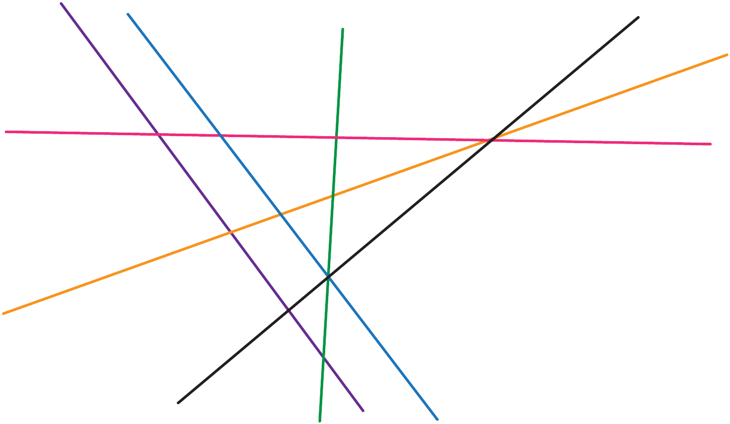


~~1~~ 2 ~~3~~ ~~4~~ ~~5~~ ~~6~~

Can straightness be forced?

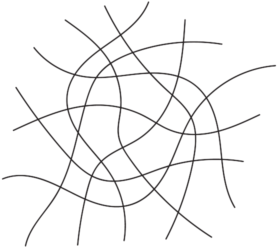


Can straightness be forced?



Can straightness be forced?

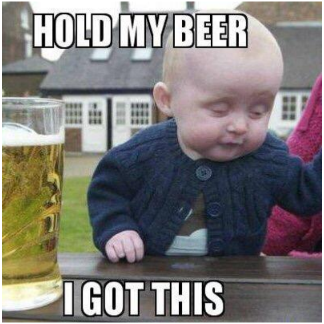
No!  Ringel in 1955 showed that there is no way to straighten the following:



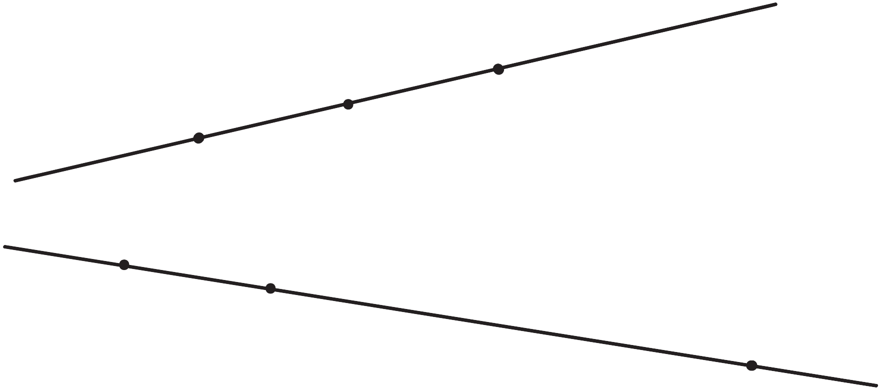
In other words, some set of lines can never be straightened!

Combinatorialists in the 1900's: We're so smart for finding a set of lines that can't be straightened!

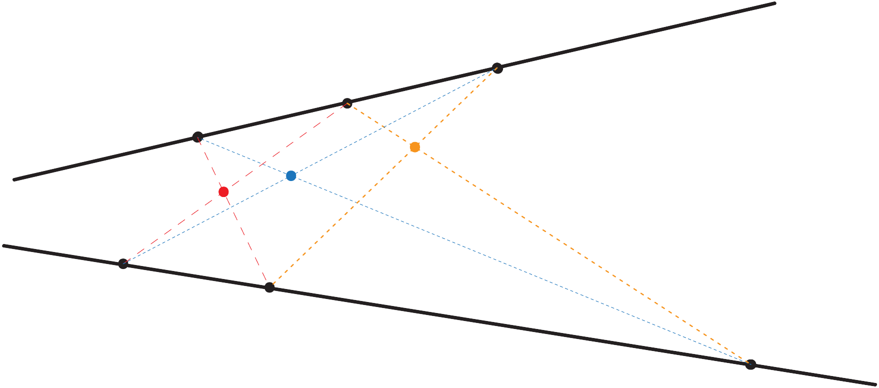
Pappus in the 300's:



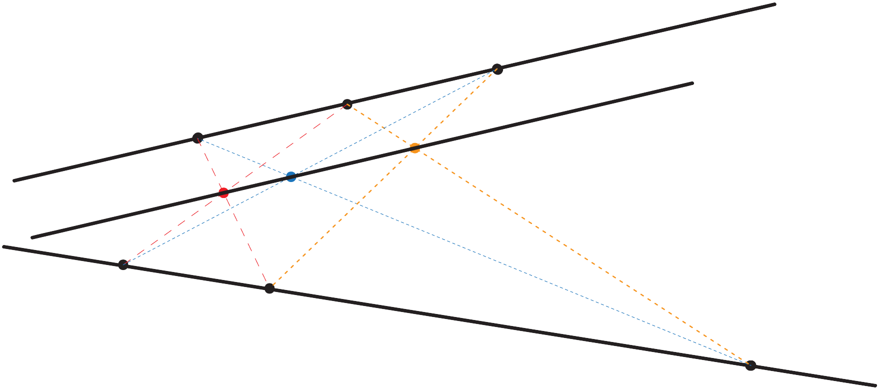
Pappus has the last laugh



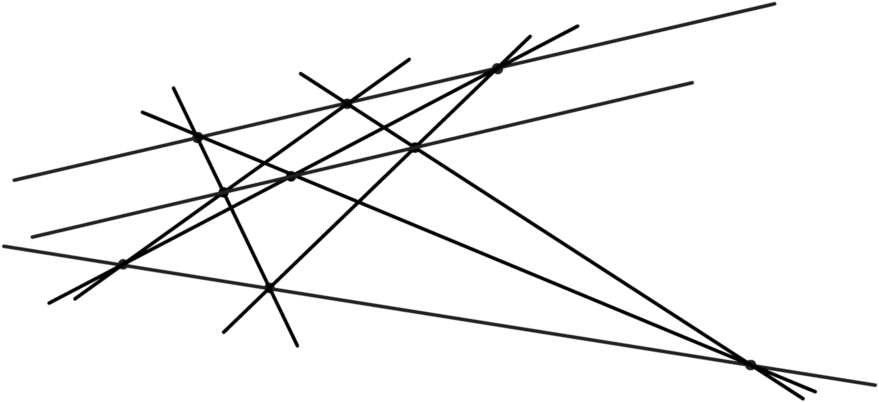
Pappus has the last laugh



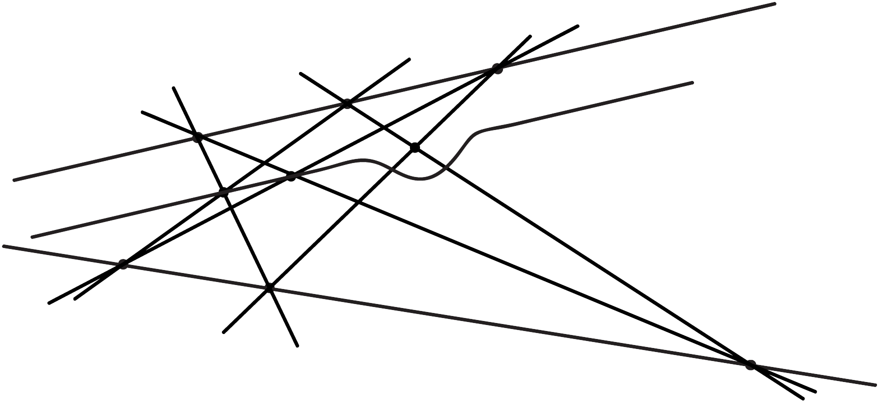
Pappus has the last laugh



Pappus has the last laugh



Pappus has the last laugh



When can straightness be forced?

Conjecture (Grünbaum 1969)

Every set of lines with at most 8 *lines can be straightened.*

When can straightness be forced?

Conjecture (Grünbaum 1969)

Every set of lines with at most 8 *lines can be straightened.*

Theorem (Goodman, Pollack 1980)

If we have at most 8 *lines, we can straighten every line.*

Combinatorialization

Literally no one:

Mathematicians: OMG Let's generalize straight lines!

What we want to do:

- Find a way to encode arrangements of “non-straight hyperplanes”
- Matroids generalize linear independence (aka: normal vectors)
- We can’t use normal vectors!
- We generalize the two sides of each “non-straight hyperplane”
- Since there are two sides, we’ll call these “oriented matroids”.

Encoding points

Any point in space can be encoded with what “side” of each hyperplane that point is on.

Example

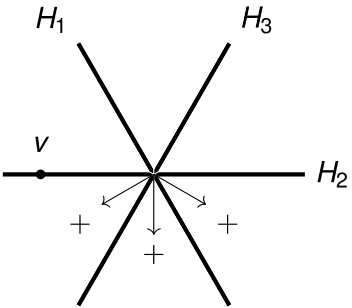
Let's look at the point v with arrangement $\mathcal{A} = \{H_1, H_2, H_3\}$.

H_1: It's on the positive side of H_1.

H_2: It's on H_2.

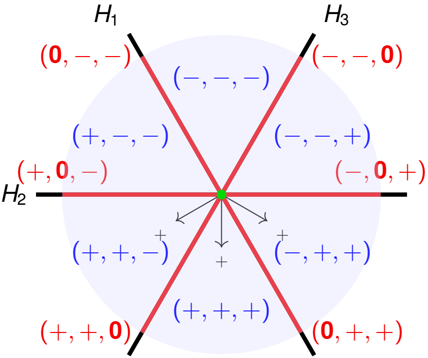
H_3: It's on the negative side of H_3.

So we can encode it as: $(+, 0, -) \in \{-, 0, +\}^{\mathcal{A}}$.



Hyperplane example

Example



Face inclusion

Next we want to construct the idea of “inclusion” so we can create a face poset.
For x and y in $\{-, 0, +\}^{\mathcal{A}}$ we define $x \circ y$ component-wise:

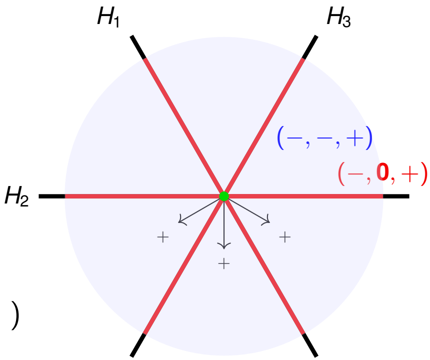
$$(x \circ y)_i = \begin{cases} x_i & \text{if } x_i \neq 0 \\ y_i & \text{otherwise} \end{cases}$$

The operation $\circ$ is known as *composition*.

Hyperplane example

For “straight” hyperplanes we have $x \subseteq y$ if and only if $x \circ y = y$.

$$(x \circ y)_i = \begin{cases} x_i & \text{if } x_i \neq 0 \\ y_i & \text{otherwise} \end{cases}$$



Example

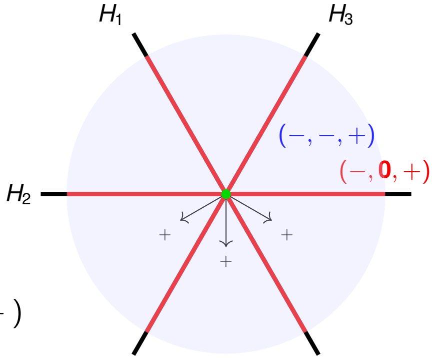
$(-,0,+) \circ (-,-,+) = (\quad,\quad,\quad)$

$(0,+,+) \circ (+,+,-) = (\quad,\quad,\quad)$

Hyperplane example

For "straight" hyperplanes we have $x \subseteq y$ if and only if $x \circ y = y$.

$$(x \circ y)_i = \begin{cases} x_i & \text{if } x_i \neq 0 \\ y_i & \text{otherwise} \end{cases}$$



Example

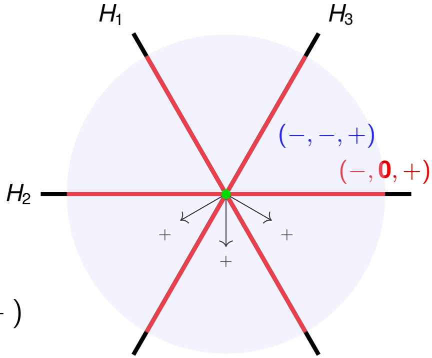
$(-,0,+) \circ (-,-,+) = (\,-\,,\quad,\,+\,)$

$(0,+,+) \circ (+,+,-) = (\quad,\quad,\quad)$

Hyperplane example

For “straight” hyperplanes we have $x \subseteq y$ if and only if $x \circ y = y$.

$$(x \circ y)_i = \begin{cases} x_i & \text{if } x_i \neq 0 \\ y_i & \text{otherwise} \end{cases}$$



Example

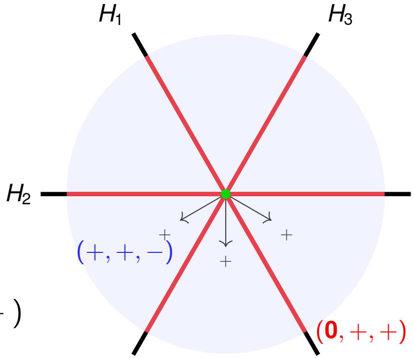
$$(-,0,+) \circ (-,-,+) = (\ -\ ,\ -\ ,\ +\ )$$

$$(0,+,+) \circ (+,+,-) = (\quad ,\quad ,\quad)$$

Hyperplane example

For "straight" hyperplanes we have $x \subseteq y$ if and only if $x \circ y = y$.

$$(x \circ y)_i = \begin{cases} x_i & \text{if } x_i \neq 0 \\ y_i & \text{otherwise} \end{cases}$$



Example

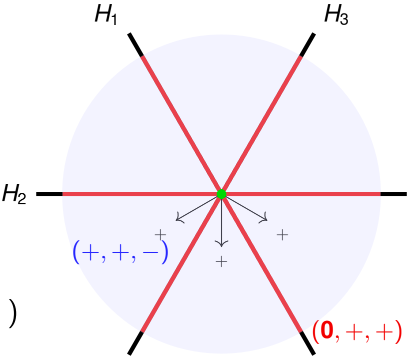
$(-,0,+) \circ (-,-,+) = (\ -\ ,\ -\ ,\ +\ )$

$(0,+,+) \circ (+,+,-) = (\quad,\quad,\quad)$

Hyperplane example

For "straight" hyperplanes we have $x \subseteq y$ if and only if $x \circ y = y$.

$$(x \circ y)_i = \begin{cases} x_i & \text{if } x_i \neq 0 \\ y_i & \text{otherwise} \end{cases}$$



Example

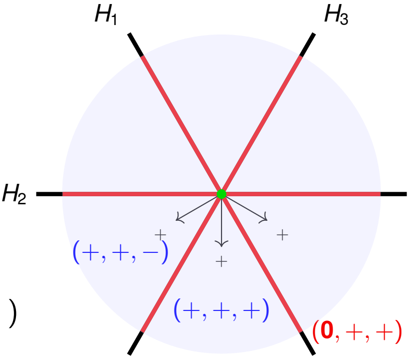
$(-,0,+) \circ (-,-,+) = (\ -\ ,\ -\ ,\ +\ )$

$(0,+,+) \circ (+,+,-) = (\quad ,\ +\ ,\ +\ )$

Hyperplane example

For “straight” hyperplanes we have $x \subseteq y$ if and only if $x \circ y = y$.

$$(x \circ y)_i = \begin{cases} x_i & \text{if } x_i \neq 0 \\ y_i & \text{otherwise} \end{cases}$$



Example

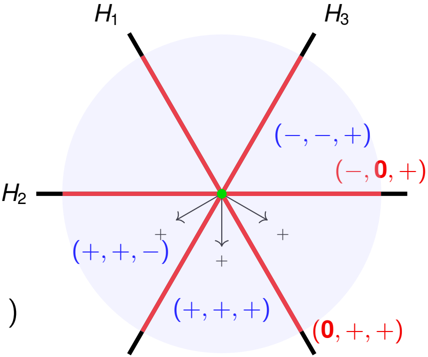
$$(-,0,+) \circ (-,-,+) = (\,-\,,\,-\,,\,+\,)$$

$$(0,+,+) \circ (+,+,-) = (\,+\,,\,+\,,\,+\,)$$

Hyperplane example

For “straight” hyperplanes we have $x \subseteq y$ if and only if $x \circ y = y$.

$$(x \circ y)_i = \begin{cases} x_i & \text{if } x_i \neq 0 \\ y_i & \text{otherwise} \end{cases}$$



Example

$(-,0,+) \circ (-,-,+) = (\,-\,,\,-\,,\,+\,)$

$(0,+,+) \circ (+,+,-) = (\,+\,,\,+\,,\,+\,)$

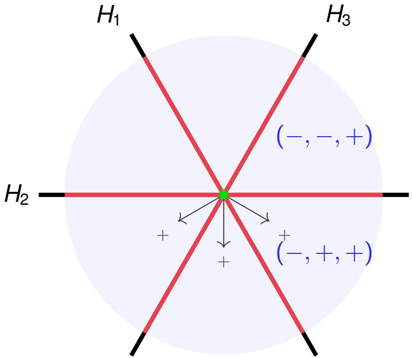
Separating faces

Next we want to see what separates faces from one another. For x and y in $\{-, 0, +\}^{\mathcal{A}}$ we define the *separation set of x and y* to be the set:

$$S(x, y) = \{H \in \mathcal{A} \mid x_H = -y_H \neq 0\}$$

Hyperplane example

$S(x, y) =$
$\{H \in \mathcal{A} \mid x_H = -y_H \neq 0\}$



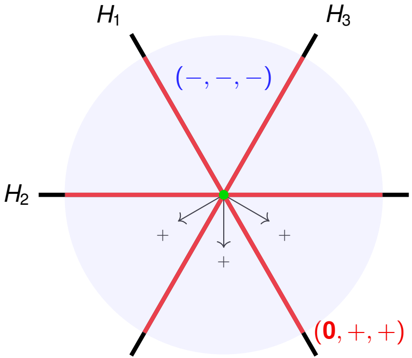
Example

$S((-,-,+),(-,+,+)) = \{H_2\}$

$S((0,+,+),(-,-,-)) = \{H_2, H_3\}$

Hyperplane example

$S(x, y) =$
$\{H \in \mathcal{A} \mid x_H = -y_H \neq 0\}$



Example

$S((-,-,+),(-,+,+)) = \{H_2\}$

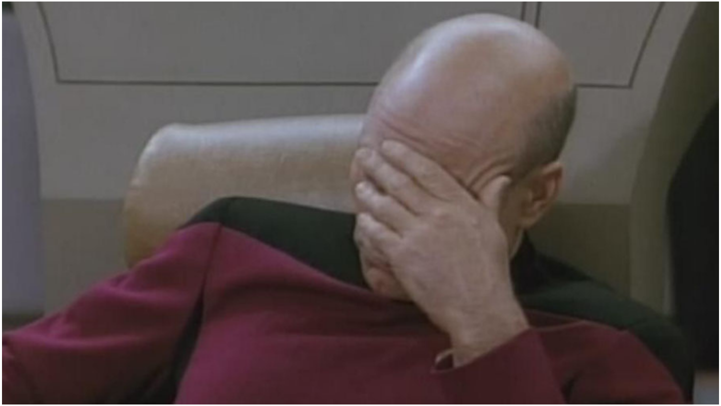
$S((0,+,+),(-,-,-)) = \{H_2, H_3\}$

An oriented definition - Covector axioms

An *oriented matroid* $M = (E, \mathcal{L})$ is a set E and a subset $\mathcal{L} \subseteq \{-, 0, +\}^E$ such that:

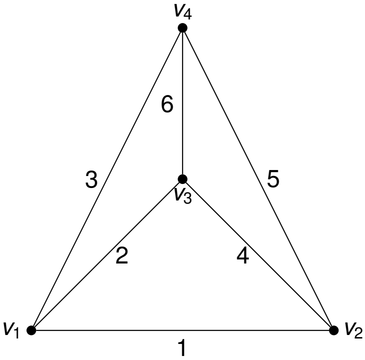
1. $(0, \ldots, 0) \in \mathcal{L}$.
2. If $x \in \mathcal{L}$ then $-x \in \mathcal{L}$.
3. If $x, y \in \mathcal{L}$ then $x \circ y \in \mathcal{L}$.
4. If $x, y \in \mathcal{L}$ and $e \in S(x, y)$ then there exists $z \in \mathcal{L}$ such that $z_e = 0$ and $z_f = (x \circ y)_f = (y \circ x)_f$ for all $f \notin S(x, y)$.

We call $\mathcal{L}$ the *set of covectors* of the oriented matroid.



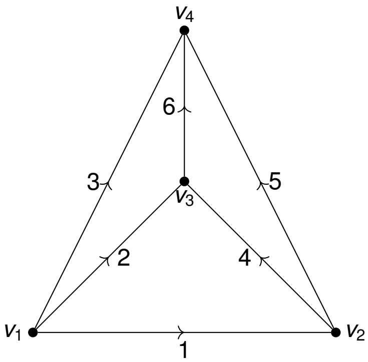
Another way

A generalization from graphical matroids (Las Vergnas).



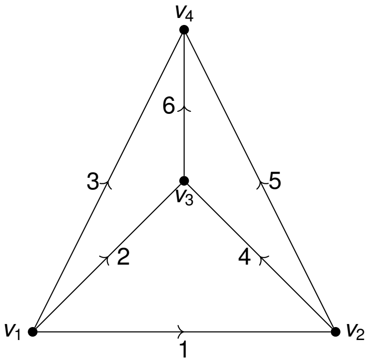
Another way

A generalization from graphical matroids (Las Vergnas).



Circuits

A *circuit* is a closed loop in a digraph (no repeated vertices/edges).

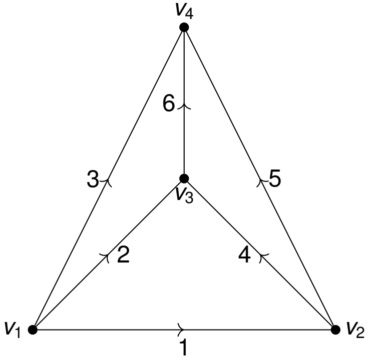


Circuits

A *circuit* is a closed loop in a digraph (no repeated vertices/edges).

We have 14 circuits:

$1, 4, -2$	$1, 4, 6, -3$
$1, 5, -3$	$1, 5, -6, -2$
$2, 6, -3$	$2, -4, 5, -3$
$4, 6, -5$	

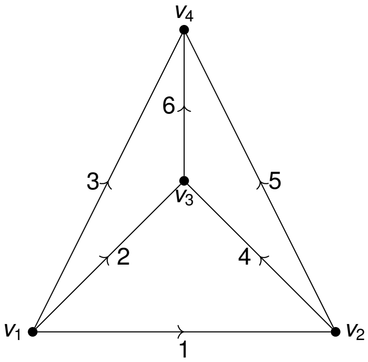


Circuits

A *circuit* is a closed loop in a digraph (no repeated vertices/edges).
We have 14 circuits:

$1, 4, -2$	$1, 4, 6, -3$
$1, 5, -3$	$1, 5, -6, -2$
$2, 6, -3$	$2, -4, 5, -3$
$4, 6, -5$	

These are called *signed subsets* as we take a subset of $[n]$ and we give a sign to each number.



Another oriented matroid - Circuit axioms

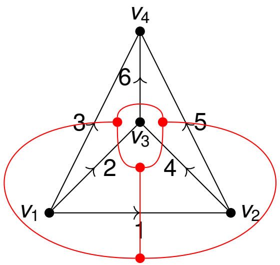
An *oriented matroid* $M = (E, \mathcal{C})$ is a set E and a collection $\mathcal{C}$ of signed subsets of E such that:

1. $\emptyset \notin \mathcal{C}$
2. $\mathcal{C} = -\mathcal{C}$
3. For all $x, y \in \mathcal{C}$ if x and y have the same underlying set then either $x = y$ or $x = -y$.
4. For all $x, y \in \mathcal{C}$, $x \neq -y$ and $y \in X^+ \cap Y^-$ there is a $z \in \mathcal{C}$ such that:
   - $Z^+ \subseteq (x^+ \cup y^+) \setminus \{e\}$
   - $Z^- \subseteq (x^- \cup y^-) \setminus \{e\}$

We call $\mathcal{C}$ the *set of circuits* of the oriented matroid.

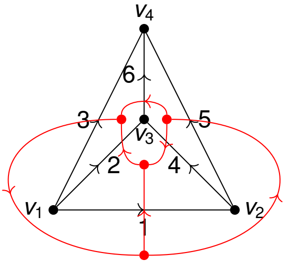
Cocircuits

Using circuits, we can also "dualize" our oriented matroids by looking at the dual graph!



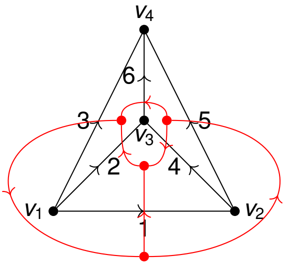
Cocircuits

Using circuits, we can also "dualize" our oriented matroids by looking at the dual graph!

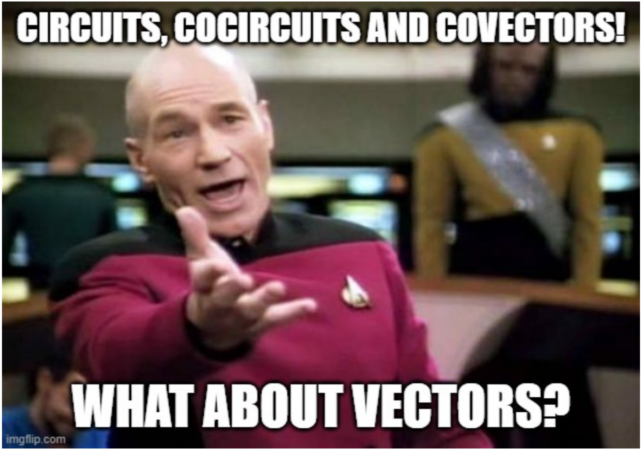


Cocircuits

Using circuits, we can also “dualize” our oriented matroids by looking at the dual graph!

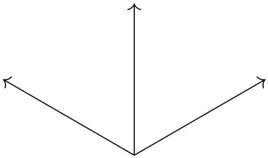


The circuits in the dual graph are called *cocircuits* $\mathcal{C}^\star$ (of the oriented matroid $(E, \mathcal{C})$).
Fun fact: these cocircuits also define an oriented matroid!


CIRCUITS, COCIRCUITS AND COVECTORS!
WHAT ABOUT VECTORS?
imgflip.com

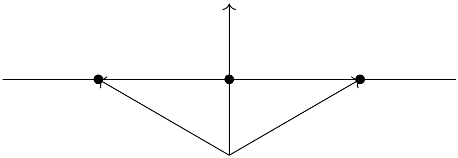
Affine point configurations

Start with a set of vectors V in $\mathbb{R}^n \setminus \{0\}$ and look at what they look like in affine space.



Affine point configurations

Start with a set of vectors V in $\mathbb{R}^n \setminus \{0\}$ and look at what they look like in affine space.



Affine point configurations

Start with a set of vectors V in $\mathbb{R}^n \setminus \{0\}$ and look at what they look like in affine space.

Radon partitions

Radon partition:
$\mathrm{int}(A) \cap \mathrm{int}(B) \neq \emptyset$

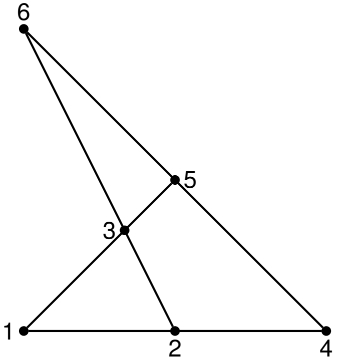
Example

$\{1, 4\}, \{2\}$

$\{2, 5\}, \{3, 4\}$

$\{1, 3, 5\}, \{4, 6\}$

$\{1, 2, 4, 6\}, \{3, 5\}$



Radon partitions

Radon partition:
$\text{int}(A) \cap \text{int}(B) \neq \emptyset$

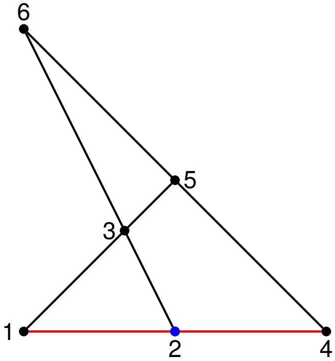
Example

$\{1,4\},\{2\}$

$\{2,5\},\{3,4\}$

$\{1,3,5\},\{4,6\}$

$\{1,2,4,6\},\{3,5\}$



Radon partitions

Radon partition:
$\text{int}(A) \cap \text{int}(B) \neq \emptyset$

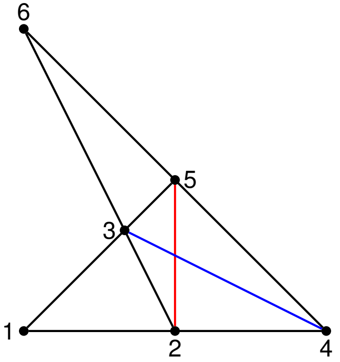
Example

$\{1, 4\}, \{2\}$

$\{2, 5\}, \{3, 4\}$

$\{1, 3, 5\}, \{4, 6\}$

$\{1, 2, 4, 6\}, \{3, 5\}$



Radon partitions

Radon partition:
$\text{int}(A) \cap \text{int}(B) \neq \emptyset$

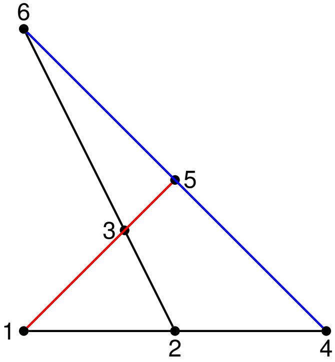
Example

$\{1, 4\}, \{2\}$

$\{2, 5\}, \{3, 4\}$

~~$\{1, 3, 5\}, \{4, 6\}$~~

$\{1, 2, 4, 6\}, \{3, 5\}$



Radon partitions

Radon partition:
$\text{int}(A) \cap \text{int}(B) \neq \emptyset$

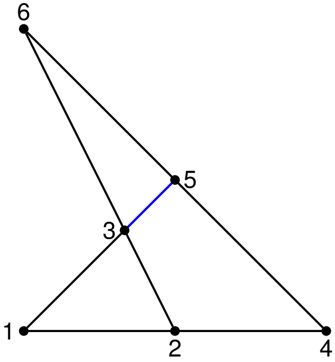
Example

$\{1,4\},\{2\}$

$\{2,5\},\{3,4\}$

~~$\{1,3,5\},\{4,6\}$~~

$\{1,2,4,6\},\{3,5\}$



Radon partitions, part 2

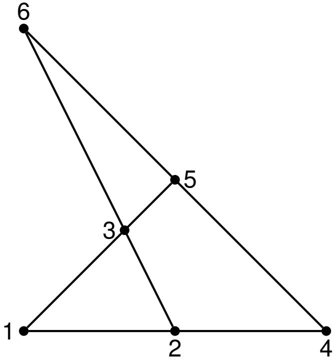
Radon partition:
$\mathrm{int}(A) \cap \mathrm{int}(B) \neq \emptyset$

Example

$\{1, 4, -2\}$

$\{2, 5, -3, -4\}$

$\{1, 2, -3, 4, -5, 6\}$



Another oriented matroid - Vector axioms

An *oriented matroid $M = (E, \mathcal{V})$* is a set E and a collection $\mathcal{V}$ of signed subsets of E such that:

1. $\emptyset \in \mathcal{V}$
2. $\mathcal{V} = -\mathcal{V}$
3. For all $x, y \in \mathcal{V}$, $x \circ y \in \mathcal{V}$
4. For all $x, y \in \mathcal{V}$, $e \in X^+ \cap Y^-$ and $f \in (\underline{x} \backslash \underline{y}) \cup (\underline{y} \backslash \underline{x}) \cup (x^+ \cap y^+) \cup (x^- \cap y^-)$ there is a $z \in \mathcal{V}$ such that:
   - $Z^+ \subseteq (x^+ \cup y^+) \setminus \{e\}$
   - $Z^- \subseteq (x^- \cup y^-) \setminus \{e\}$
   - and $f \in \underline{z}$.

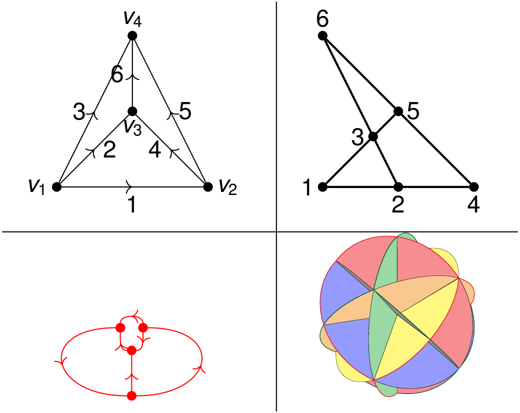
We call $\mathcal{V}$ the *set of vectors* of the oriented matroid.

They're all the same!

All of these ways of defining an oriented matroid are the same!
In other words, M is an oriented matroid

- iff M has a set of covectors $\mathcal{L}$ satisfying covector axioms
- iff M has a set of circuits $\mathcal{C}$ satisfying cicruit axioms
- iff M has a set of cocircuits $\mathcal{C}^\star$ satisfying circuit axioms
- iff M has a set of vectors $\mathcal{V}$ satisfying vector axioms

Example



(image by Vincent Pilaud)


BUT WAIT
THERE'S MORE

That's not all!

There are even MORE ways to define oriented matroids! We can use:

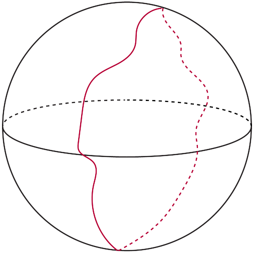
- Vector configurations
- Vector subspaces
- Linear programming
- Chirotopes (molecular chemistry)
- Allowable sequences
- And more!!!

$\mathbb{R}$y str8

A spherical approach

Restrict ourselves to spheres S^d.

- S^d is the d-dim sphere which lives in $\mathbb{R}^{d+1}$.
- A $(d-1)$-*subsphere* is $S \subseteq S^d$ and $S \simeq S^{d-1}$.



Theorem (Jordan-Brouwer separation theorem)

Let S be a $(d-1)$-subsphere in $\mathbb{R}^{d+1}$. Then $S' = S^d \backslash S$ consists of exactly two connected components. The set S is their common boundary.

Fake spheres

A *pseudosphere* is a $(d-1)$-subsphere equivalent to the equator. (Alexander horned sphere)
An *arrangement of pseudospheres S_E* is:

- $S_A = \cap_{e \in A} S_e$ is a sphere for all $A \subseteq E$.
- If $S_A \not\subseteq S_e$ for $A \subseteq E$, $e \in E$, and S_e^+ and S_e^- are two sides of S_e, then $S_A \cap S_e$ is a pseudosphere in S_A with sides $S_A \cap S_e^+$ and $S_A \cap S_e^-$.

A *signed* arrangement of pseudosperes S_E is when S_e^+ is "positive" and S_e^- is "negative".

Topological representation theorem

Theorem (Folkman-Lawrence 1978; Edmonds-Mandel 1982)

Let $\mathcal{L}$ be a set of signed vectors. The following are equivalent:

- *$\mathcal{L}$ is the set of covectors of a (loop-free) oriented matroid of rank $d+1$.*
- *$\mathcal{L}$ is the set of covectors of a signed arrangement $\mathcal{A}$ of pseudospheres in S^d (which is essential, centrally symmetric and whose cell complex is shellable)*

(
- *loop-free*: There is no $e \in E$ such that $x_e = 0$ for all $x \in \mathcal{L}$.
- *essential*: $\cap S_E = \emptyset$
- *centrally symmetric*: $-S_e = S_e$ for all $e \in E$.
- *shellable*: no room on slide.

)

Topological representation theorem

Theorem (Folkman-Lawrence 1978; Edmonds-Mandel 1982)

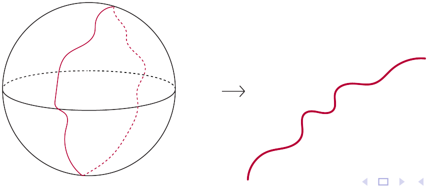
(nice) Oriented matroids are equivalent to (nice) signed pseudosphere arrangements.

Lines

Theorem (Folkman-Lawrence 1978; Edmonds-Mandel 1982)

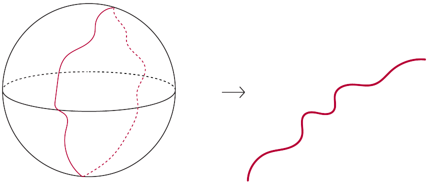
Let $\mathcal{L}$ be a set of signed vectors. The following are equivalent:

- *$\mathcal{L}$ is the set of covectors of a (loop-free) oriented matroid of rank* 3.
- *$\mathcal{L}$ is the set of covectors of a signed arrangement $\mathcal{A}$ of pseudospheres in S^2 (which is essential, centrally symmetric and whose cell complex is shellable)*



Lines

Theorem (Folkman-Lawrence 1978; Edmonds-Mandel 1982)
Oriented matroids of rank 3 *are equivalent to sets of lines.*



How about stretching?

An oriented matroid is *realizable* if we can “straighten” all the pseudospheres into spheres.

Theorem

An oriented matroid is realizable if and only if it is the oriented matroid of a hyperplane arrangement.

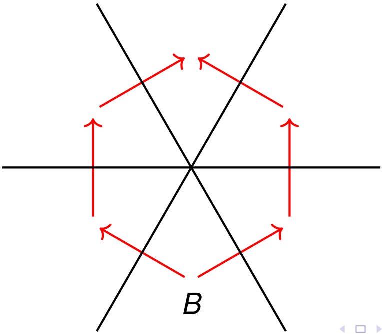
An unsolved problem
(One of 50+)

Poset of regions

Let B be a fixed region called the *base region*.

The *poset of regions* is the set $\mathscr{R}$ of regions with the partial order:

$$R \leq R' \iff S(B,R) \subseteq S(B,R')$$

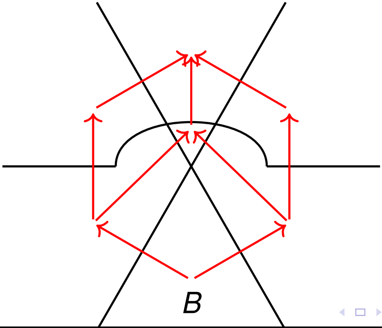


Poset of topes

A *tope* is a maximal covector in an oriented matroid M. Let B be a fixed tope called the *base tope*.

The *poset of topes* is the set of all topes with the partial order:

$$T \leq T' \iff S(B,T) \subseteq S(B,T')$$

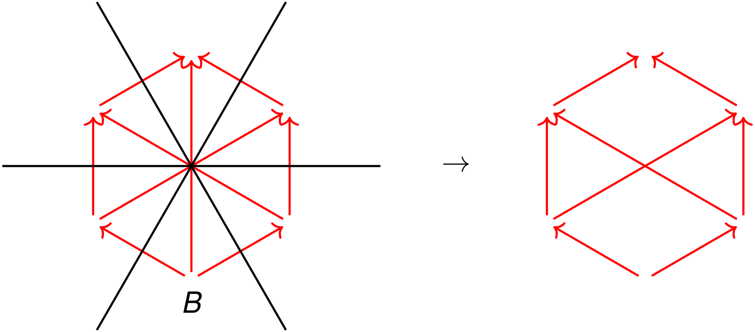


Strong order (aka Bruhat order)

Let B be a fixed region called the *base region*.

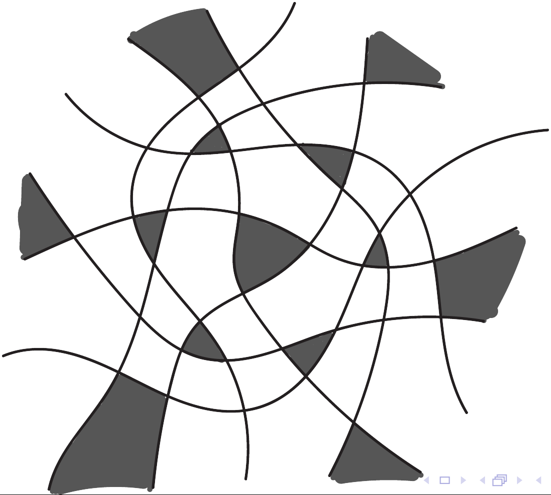
The *strong order* is the set $\mathscr{R}$ with the order:

$$R \leq R' \iff |S(B,R)| < |S(B,R')| \text{ and } R = \rho_H(R') \text{ for some } H \in \mathcal{A}$$



Strong order (aka Bruhat order)

A solved problem

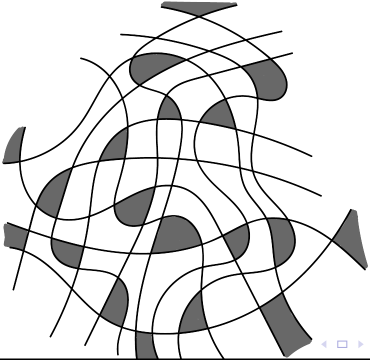


A solved problem

Conjecture (Grünbaum 1969)

All sets of straight lines have 2 *triangles with a common vertex.*

- Disproven by Ljubić, Roudneff and Sturmfels in 1989:




IGHT IMMA HEAD OUT