The facial weak order

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York University

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Outline

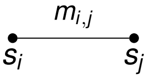
- A tale of two stories:
  - Grouping reflections.
  - Arranging hyperplanes.
- The facial weak order in all its glory.
- Yeah, but is it a lattice? And other fun questions.
- Current research

Coxeter systems

- *Finite Coxeter System* (W, S) such that

$$W := \langle s \in S \mid (s_i s_j)^{m_{i,j}} = e \text{ for } s_i, s_j \in S \rangle$$

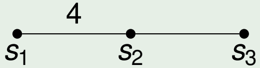
  where $m_{i,j} \in \mathbb{N}^\star$ and $m_{i,j} = 1$ only if $i = j$.
- A *Coxeter diagram* Γ_W for a Coxeter System (W, S) has S as a vertex set and an edge labelled $m_{i,j}$ when $m_{i,j} > 2$.



Example

$$W_{B_3} = \left\langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^4 = (s_2 s_3)^3 = (s_1 s_3)^2 = e \right\rangle$$

Γ_{B_3} :



Coxeter systems

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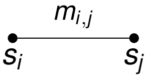
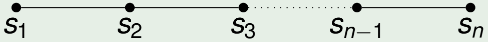


Example

$W_{A_n} = S_{n+1}$, symmetric group.

Γ_{A_n} :



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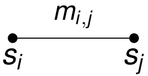
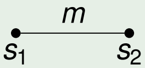


Example

$W_{I_2(m)} = \mathcal{D}(m)$, dihedral group of order $2m$.

$\Gamma_{I_2(m)}$:



Weak order

Let (W, S) be a Coxeter system.

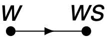
- Let $w \in W$ such that $w = s_1 \dots s_n$ for some $s_i \in S$. We say that w has *length* n, $\ell(w) = n$, if n is minimal.

Example

Let Γ_{A_2} : 

.

$\ell(stst) = 2$ as $stst = tstt = ts$.

- Let the *(right) weak order* be the order $\leq_R$ on the Cayley graph where 

and $\ell(w) < \ell(ws)$.

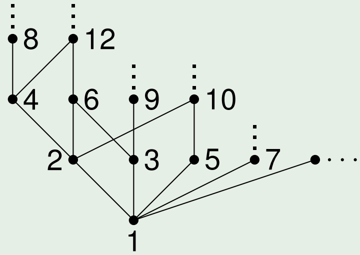
Lattice

- *Lattice* - poset where every two elements have a *meet* (greatest lower bound) and *join* (least upper bound).

Example

The lattice $(\mathbb{N}, |)$ where $a \leq b \iff a \mid b$.

- meet - greatest common divisor
- join - least common multiple



Weak order lattice

Theorem (Björner '84)

Let (W, S) be a finite Coxeter system. The weak order is a lattice.

- For finite Coxeter systems, there exists a longest element in the weak order, $w_\circ$.

Example

Let Γ_{A_2} : 

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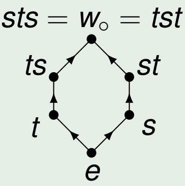
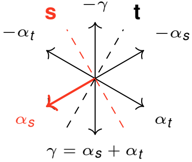


Root System

- Let $(V, \langle \cdot, \cdot \rangle)$ be a real Euclidean space.
- Let W be a group generated by a set of reflections S. $W \hookrightarrow O(V)$ gives representation as a finite reflection group.
- The reflection associated to $\alpha \in V \backslash \{0\}$ is

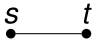
$$s_\alpha(v) = v - \frac{2\langle v, \alpha \rangle}{||\alpha||^2}\alpha \quad (v \in V)$$

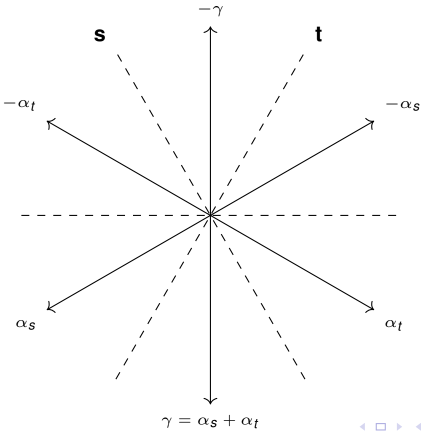


- A *root system* is $\Phi := \{\alpha \in V \mid s_\alpha \in W, ||\alpha|| = 1\}$
- We have $\Phi = \Phi^+ \sqcup \Phi^-$ decomposable into positive and negative roots.

Root systems $\leftrightarrow$ Coxeter systems

$$W_{A_2} = \left\langle s, t \;\middle|\; s^2 = t^2 = (st)^3 = e \right\rangle \quad \Gamma_{A_2} :$$





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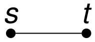

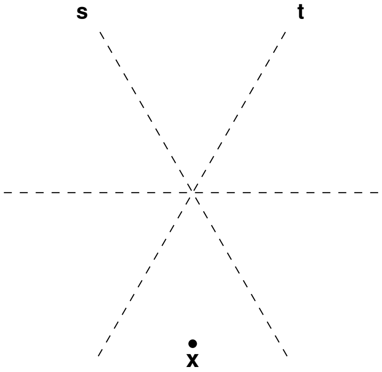

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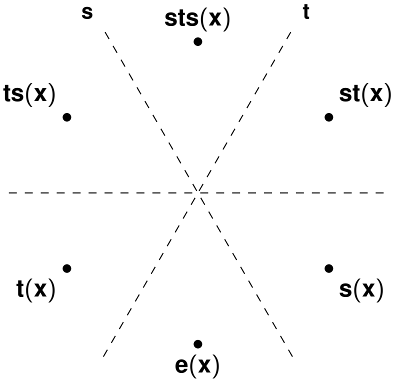

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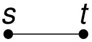
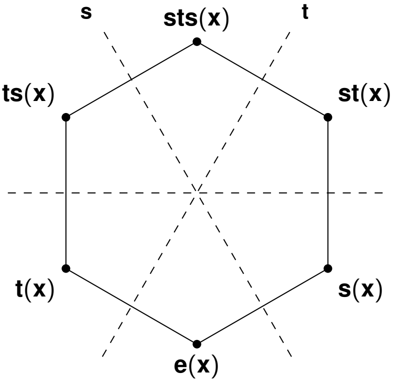


$$\mathrm{Perm}(W) = \mathrm{conv}\left\{ w(x) \;\middle|\; w \in W \right\}$$

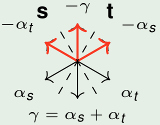


Inversion Sets

Let (W, S) be a Coxeter system.
Define *(left) inversion sets* as the set $\mathbf{N}(w) := \Phi^+ \cap w(\Phi^-)$.

Example

Let Γ_{A_2} : s •——• t , with Φ given by the roots



$$\begin{aligned}\mathbf{N}(ts) &= \Phi^+ \cap ts(\Phi^-)\\ &= \Phi^+ \cap \{\alpha_t, \gamma, -\alpha_s\}\\ &= \{\alpha_t, \gamma\}\end{aligned}$$

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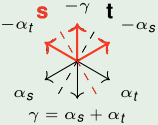

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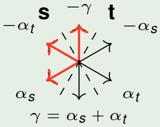

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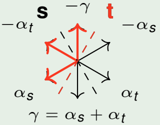

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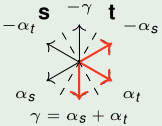
, with Φ given by the roots 

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Weak order and Inversion sets

Given $w, u \in W$ then $w \leq_{\mathrm{R}} u$ if and only if $\mathbf{N}(w) \subseteq \mathbf{N}(u)$.

Example

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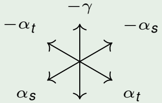
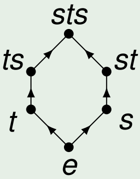
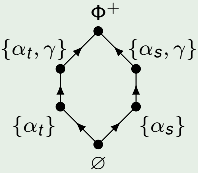
, with Φ given by the roots 

$\gamma = \alpha_s + \alpha_t$





Weak order and inversion sets

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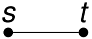
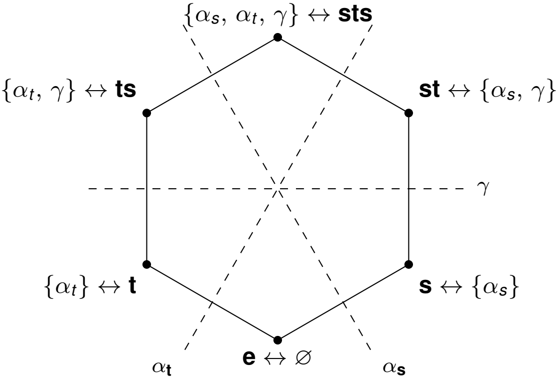
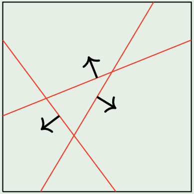




Hyperplanes

- $(V, \langle \cdot, \cdot \rangle)$ - n-dim real Euclidean vector space.
- A *hyperplane H* is $\mathrm{codim}\ 1$ subspace of V with normal e_H.

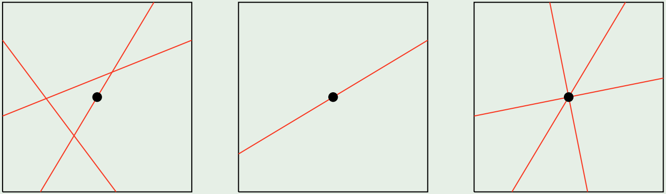
Example

Arrangements

- A *hyperplane arrangement* is $\mathcal{A} = \{H_1, H_2, \ldots, H_k\}$.
- $\mathcal{A}$ is *central* if $\{0\} \subseteq \bigcap \mathcal{A}$.
- $\mathcal{A}$ is *essential* if $\operatorname{span}\{e_H\}_{H \in \mathcal{A}} = V$.
- $\mathcal{A}$ Central & Essential $\Rightarrow \{0\} = \bigcap \mathcal{A}$.

Example



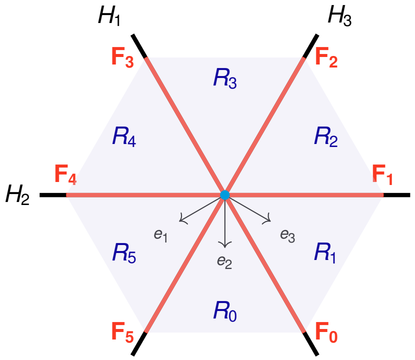
Not central

Central
Not essential

Central
Essential

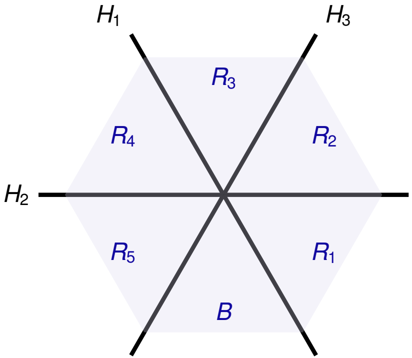
Regions and faces

- *Regions* $\mathscr{R}_{\mathcal{A}}$ - connected components of V without $\mathcal{A}$.
- *Faces* $\mathscr{F}_{\mathcal{A}}$ - intersections of closures of some regions.



Poset of regions

- *Base region* $B \in \mathscr{R}_{\mathcal{A}}$ - some fixed region
- *Separation set for* $R \in \mathscr{R}_{\mathcal{A}}$
  $S(R) := \{H \in \mathcal{A} \mid H \text{ separates } R \text{ from } B\}$



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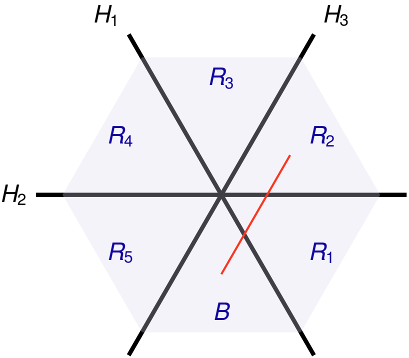

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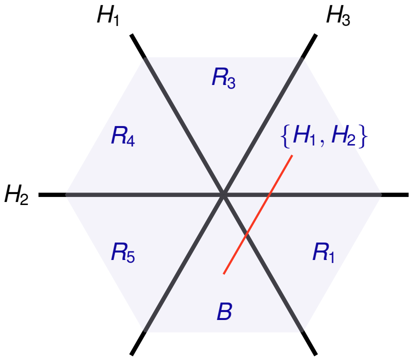

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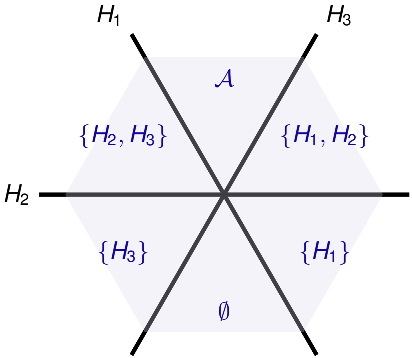

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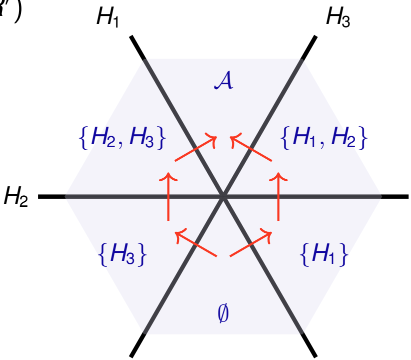
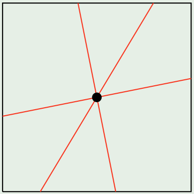
  $S(R) := \{H \in \mathcal{A} \mid H \text{ separates } R \text{ from } B\}$
- *Poset of regions* $\mathrm{PR}(\mathcal{A}, B)$ where
  $R \leq_{\mathrm{PR}} R' \iff S(R) \subseteq S(R')$



Simplicial arrangements

- A region R is *simplicial* if normal vectors for boundary hyperplanes are linearly independent.
- $\mathcal{A}$ is *simplicial* if all $\mathscr{R}_{\mathcal{A}}$ simplicial.

Example



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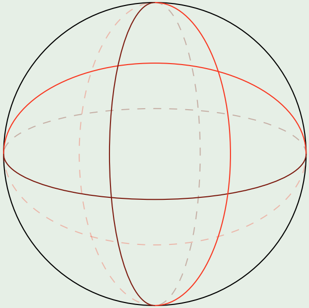


Not simplicial

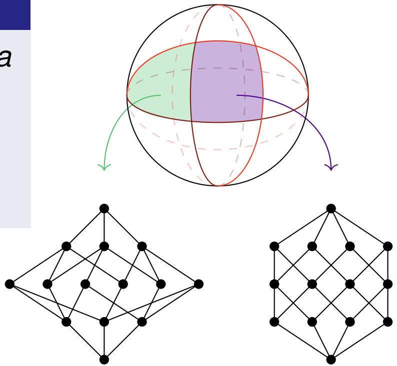
Lattice of regions

An arrangement $\mathcal{A}$ in $\mathbb{R}^n$ is *simplicial* if every region is simplicial (*i.e.*, has n boundary hyperplanes).

Theorem (Björner, Edelman, Ziegler '90)

If $\mathcal{A}$ is simplicial then $\mathrm{PR}(\mathcal{A}, B)$ is a lattice for any $B \in \mathscr{R}_{\mathcal{A}}$.

If $\mathrm{PR}(\mathcal{A}, B)$ is a lattice then B is simplicial.



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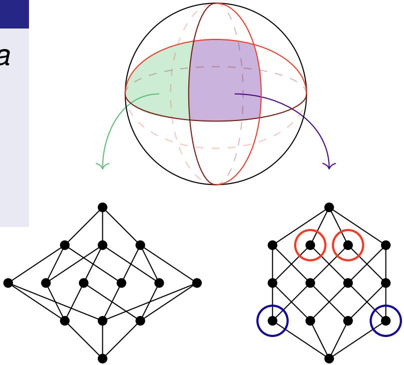


Coxeter Arrangements

Example

A *Coxeter arrangement* is the (simplicial) hyerplane arrangement associated to a Coxeter group.

Coxeter Groups		**Hyperplane Arrangements**
Reflecting hyperplanes	$\leftrightarrow$	Hyperplane arrangement
Root system	$\leftrightarrow$	Normals to hyperplanes
Inversion sets	$\leftrightarrow$	Seperation sets
Weak order	$\leftrightarrow$	Poset of regions

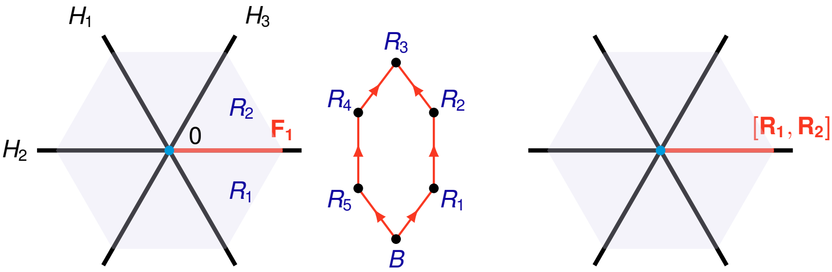
Motivation

- **2001:** Krob, Latapy, Novelli, Phan, and Schwer extended the weak order of type A Coxeter groups to all the faces of its associated arrangement. They
  - gave a local definition of this order using covers,
  - gave a global definition of this order combinatorially, and
  - showed that the poset for this order is a lattice.
- **2006:** Palacios and Ronco extended this new order to Coxeter arrangements of all types using cover relations.
- **Our Questions:**
  - Can we give a global equivalent definition to Palacios, Ronco cover relation definition?
  - What happens in the hyperplane arrangement story?
  - When is this a lattice?

Facial intervals

Proposition (Björner, Las Vergas, Sturmfels, White, Ziegler '93)

Let $\mathcal{A}$ be central with base region B. For every $F \in \mathscr{F}_{\mathcal{A}}$ there is a unique interval $[m_F, M_F]$ in $\mathrm{PR}(\mathcal{A}, B)$ such that
$$[m_F, M_F] = \left\{ R \in \mathscr{R}_{\mathcal{A}} \;\middle|\; F \subseteq \overline{R} \right\}$$



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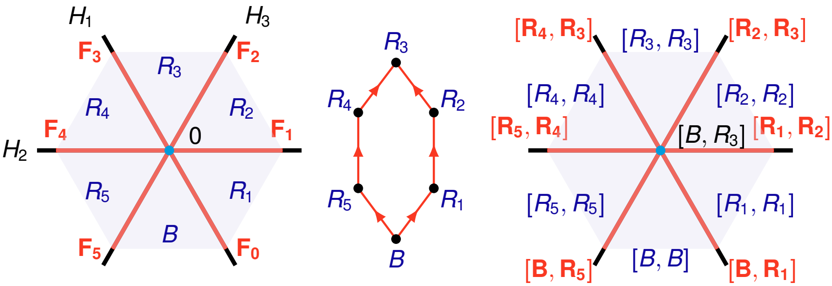
Let $\mathcal{A}$ be central with base region B. For every $F \in \mathscr{F}_{\mathcal{A}}$ there is a unique interval $[m_F, M_F]$ in $\mathrm{PR}(\mathcal{A}, B)$ such that

$$[m_F, M_F] = \left\{ R \in \mathscr{R}_{\mathcal{A}} \ \middle|\ F \subseteq \overline{R} \right\}$$



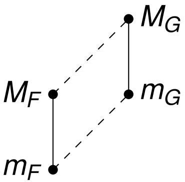
Facial weak order

Let $\mathcal{A}$ be a central hyperplane arrangement and B a base region in $\mathscr{R}_{\mathcal{A}}$.

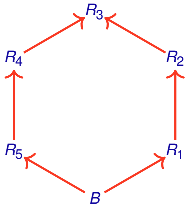
Definition

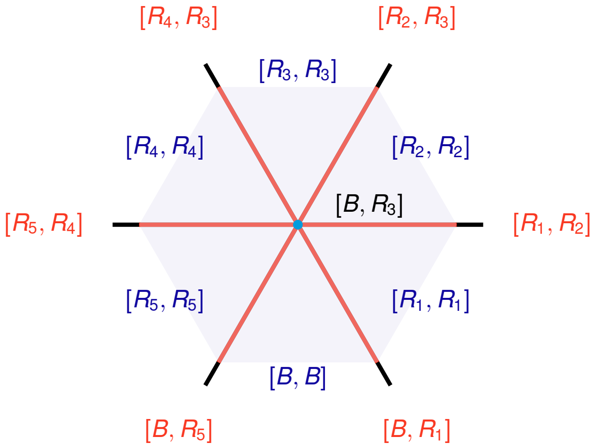
The *facial weak order* is the order $\mathrm{FW}(\mathcal{A}, B)$ on $\mathscr{F}_{\mathcal{A}}$ where for $F, G \in \mathscr{F}_{\mathcal{A}}$:

$$F \leq_{\mathrm{F}} G \iff m_F \leq_{\mathrm{PR}} m_G \text{ and } M_F \leq_{\mathrm{PR}} M_G$$

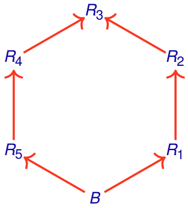


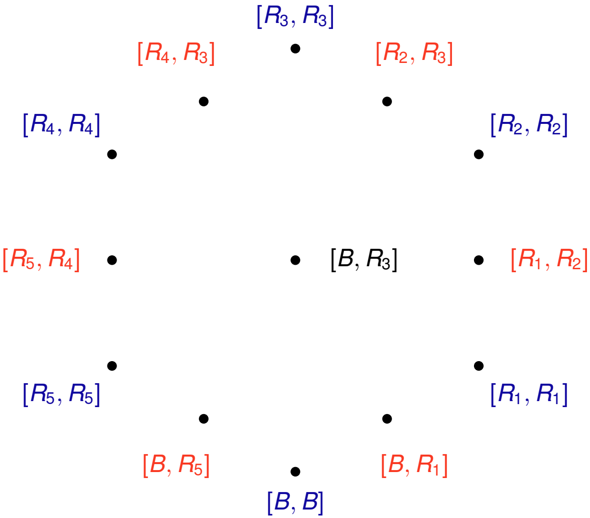
Facial weak order - Example



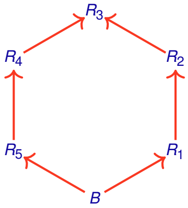


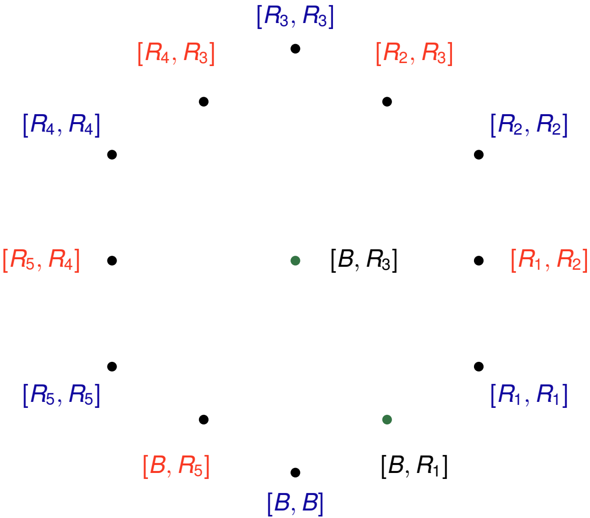
Facial weak order - Example



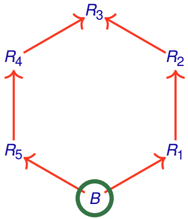


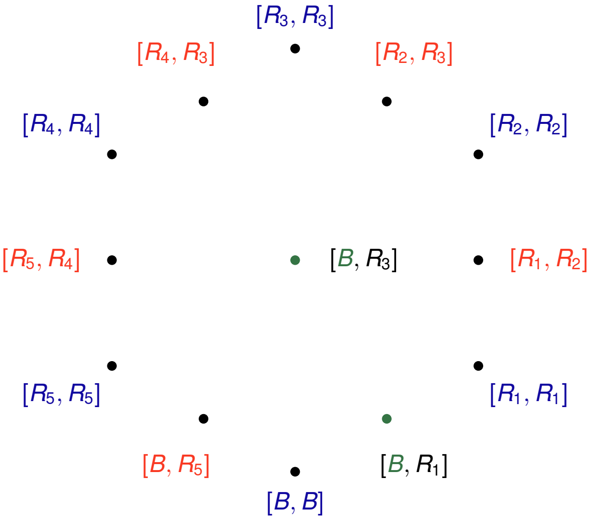
Facial weak order - Example



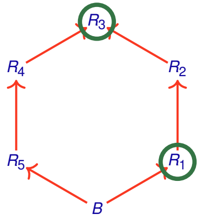


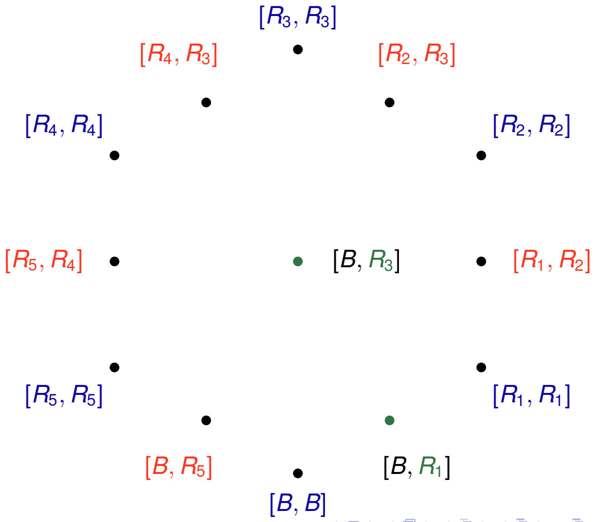
Facial weak order - Example



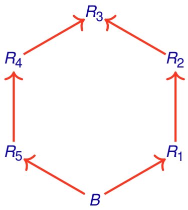


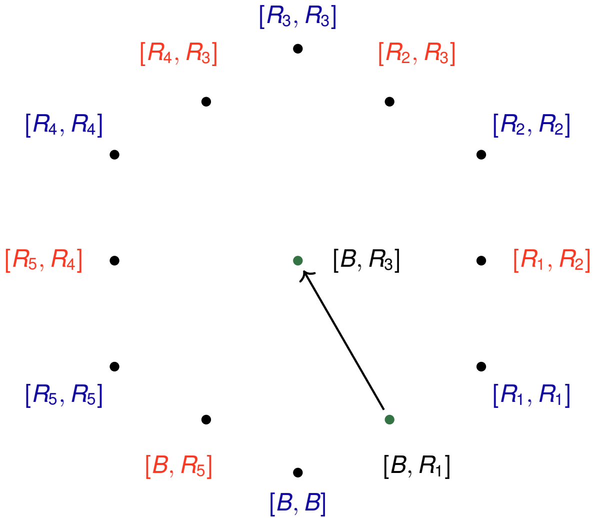
Facial weak order - Example



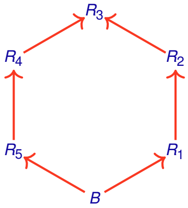


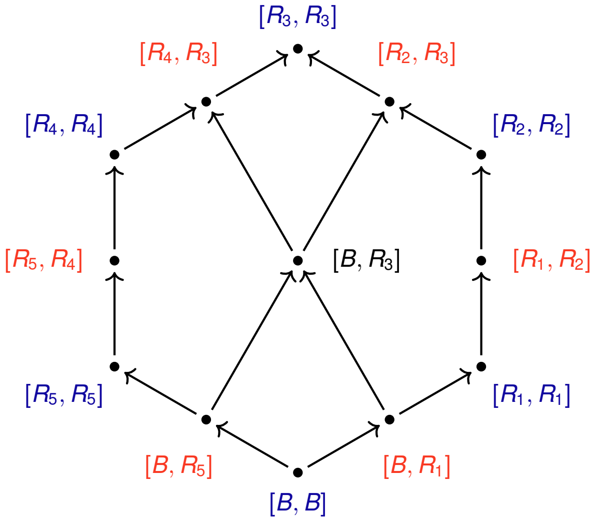
Facial weak order - Example





Facial weak order - Example





Parabolic subgroups

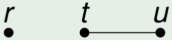
(W, S) a Coxeter system and $I \subseteq S$.

- $W_I = \langle I \rangle$ — *standard parabolic subgroup* (long elt: $w_{\circ,I}$).
- $W^I := \{w \in W \mid \ell(w) \leq \ell(ws), \text{ for all } s \in I\}$ is the set of min length coset representatives for W/W_I.
- Unique factorization: $w = w^I \cdot w_I$ with $w^I \in W^I$, $w_I \in W_I$.
- By convention in this talk xW_I means $x \in W^I$.

Example

Let Γ_W : 

and $I = \{r, t, u\}$.

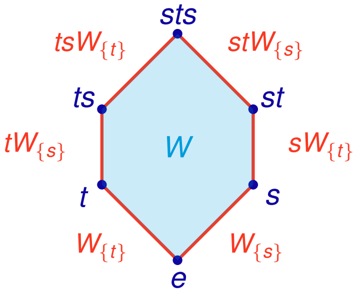
Then Γ_{W_I} : 

$$w = rtustr \qquad w = rts \cdot utr$$

Coxeter complex

(W, S) a Coxeter system and $I \subseteq S$.

- *Coxeter complex* - $\mathcal{P}_W$ - complex whose faces are all the standard parabolic cosets of W.

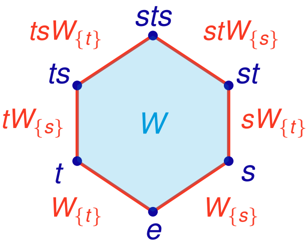


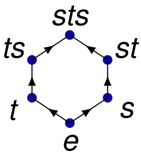
Facial intervals

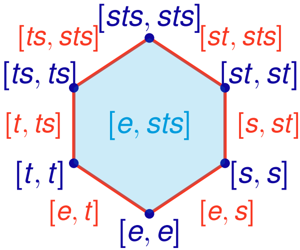
Proposition (Björner, Las Vergas, Sturmfels, White, Ziegler '93)

Let (W, S) be a Coxeter system and xW_I a standard parabolic coset. Then there exists a unique interval $[x, xw_{\circ,I}]$ in the weak order such that

$$xW_I = [x, xw_{\circ,I}].$$





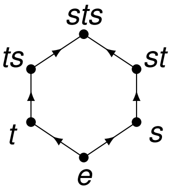


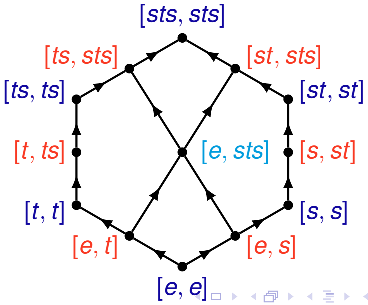
Facial weak order

Definition

Let $\leq_F$ be the order on the Coxeter complex $\mathcal{P}_W$ defined by

$$xW_I \leq_F yW_J \iff x \leq_R y \text{ and } xw_{\circ,I} \leq_R yw_{\circ,J}$$





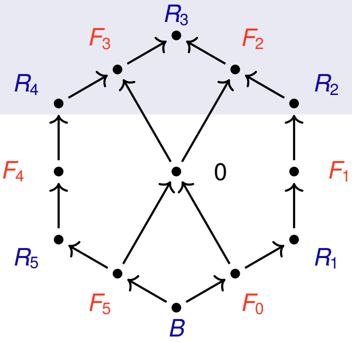
Cover relations

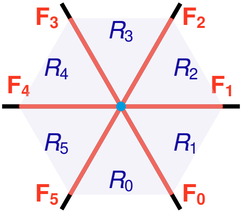
Proposition (D., Hohlweg, McConville, Pilaud, '19+)

For $F, G \in \mathscr{F}_{\mathcal{A}}$ if $|\dim(F) - \dim(G)| = 1$ and

1. $F \subseteq G$, $M_F = M_G$, *or*
2. $G \subseteq F$, $m_F = m_G$.

then $F \lessdot_{\mathrm{F}} G$.





Cover relations

Let (W, S) be a finite Coxeter system.

Definition (Krob et.al. [2001, type A], Palacios, Ronco [2006])

The *(right) facial weak order* is the order $\leq_{\mathrm{COV}}$ on the Coxeter complex $\mathcal{P}_W$ defined by cover relations of two types:

$$(1) \qquad xW_I \lessdot_{\mathrm{COV}} xW_{I\cup\{s\}} \qquad \text{if } s \notin I \text{ and } x \in W^{I\cup\{s\}},$$

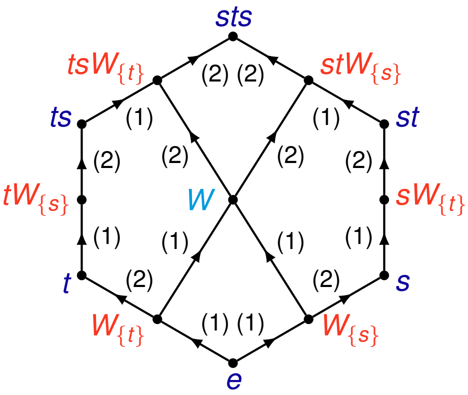
$$(2) \qquad xW_I \lessdot_{\mathrm{COV}} xw_{\circ,I}w_{\circ,I\smallsetminus\{s\}}W_{I\smallsetminus\{s\}} \qquad \text{if } s \in I,$$

where $I \subseteq S$ and $x \in W^I$.

Cover relations example

(1) $xW_I \lessdot_{\text{COV}} xW_{I \cup \{s\}}$ if $s \notin I$ and $x \in W^{I \cup \{s\}}$

(2) $xW_I \lessdot_{\text{COV}} xw_{\circ,I} w_{\circ,I \setminus \{s\}} W_{I \setminus \{s\}}$ if $s \in I$



Zonotopes

- *Zonotope $Z_{\mathcal{A}}$* is the convex polytope:

$$Z_{\mathcal{A}} := \left\{ v \in V \;\middle|\; v = \sum_{i=1}^{k} \lambda_i e_i, \text{ such that } |\lambda_i| \leq 1 \text{ for all } i \right\}$$

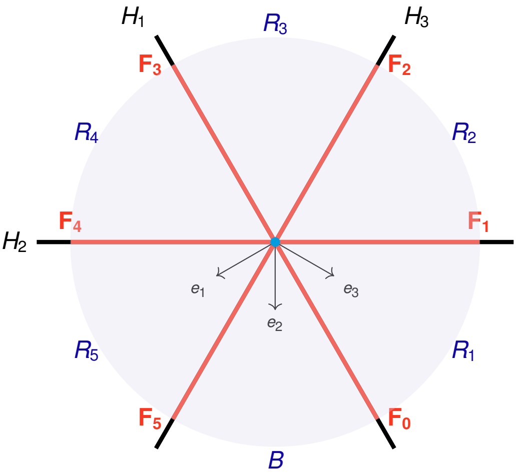
Theorem (Edelman '84, McMullen '71)

There is a bijection between $\mathscr{F}_{\mathcal{A}}$ and the nonempty faces of $Z_{\mathcal{A}}$ given by the map

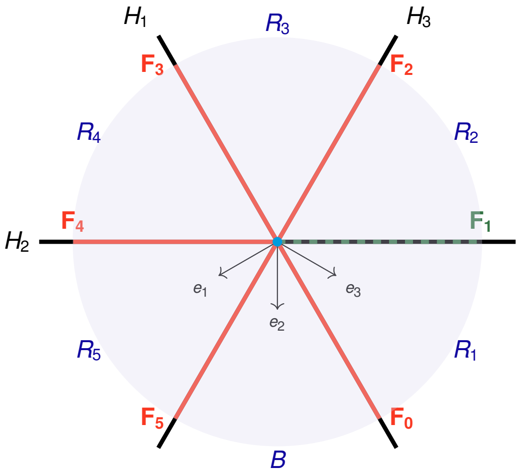
$$\tau(F) = \left\{ v \in V \;\middle|\; v = \sum_{F(H_i)=0} \lambda_i e_i + \sum_{F(H_j)\neq 0} \mu_j e_j \right\}$$

$$\textit{where } |\lambda_i| \leq 1 \textit{ for all } i \textit{ and } \mu_j = F(H_j)$$

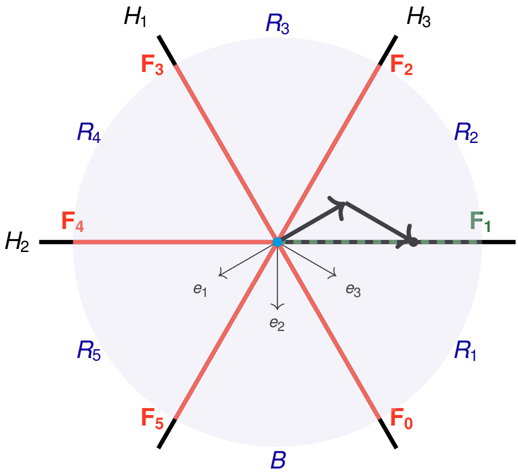
Zonotope - Construction example



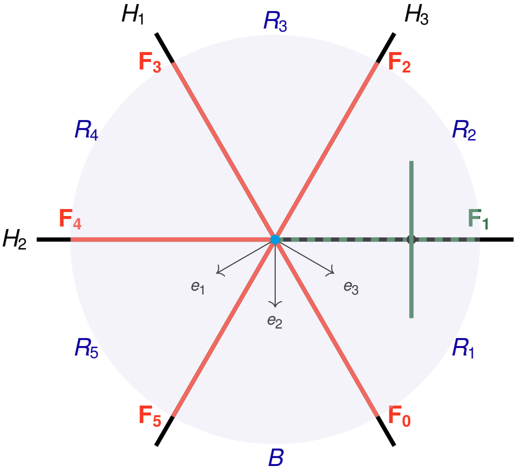
Zonotope - Construction example



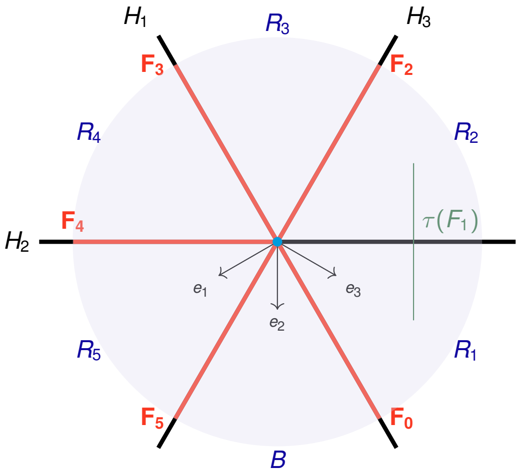
Zonotope - Construction example



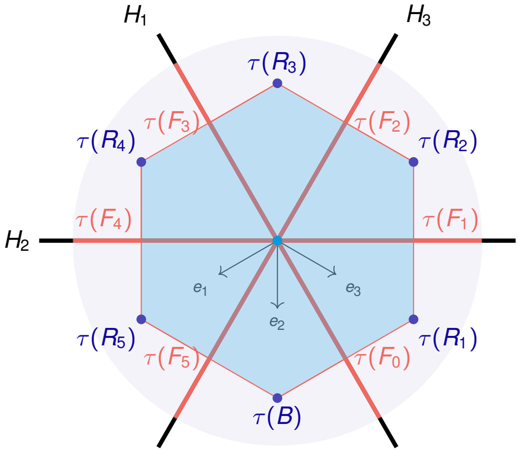
Zonotope - Construction example



Zonotope - Construction example

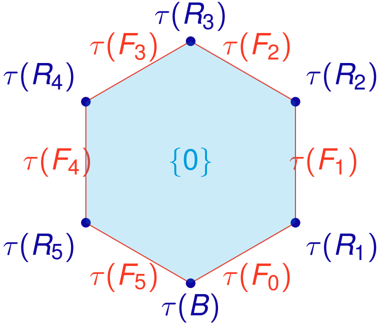


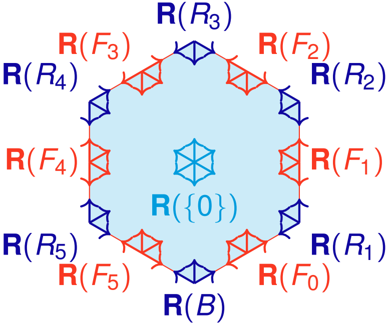
Zonotope - Construction example



Root inversion sets

- *roots* $\Phi_{\mathcal{A}} := \{\pm e_1, \pm e_2, \dots, \pm e_k\}$
- *root inversion set*
  $\mathbf{R}(F) := \{e \in \Phi_{\mathcal{A}} \mid \langle x, e \rangle \leq 0 \text{ for some } x \in F\}$.



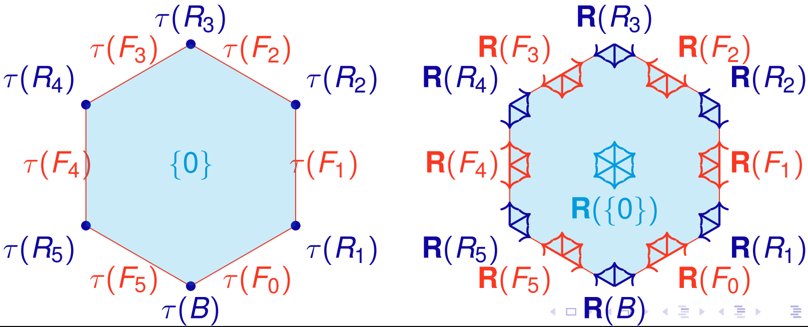


Root inversion sets

Proposition (D., Hohlweg, McConville, Pilaud '19+)

Let F be a face. Then

$$\textbf{inner primal cone}\,(\tau(F)) = \text{cone}\,(\mathbf{R}(F))\,.$$

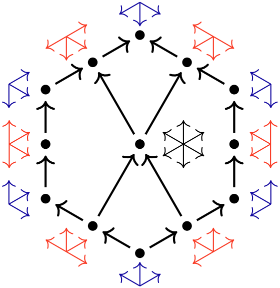


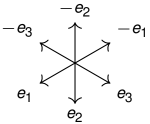
Root inversion set order

Definition

For faces F and G in $\mathscr{F}_{\mathcal{A}}$, then $F \leq_{\mathrm{RIS}} G$ if and only if

$$\mathbf{R}(F) \setminus \mathbf{R}(G) \subseteq \Phi^{-}_{\mathcal{A}} \text{ and } \mathbf{R}(G) \setminus \mathbf{R}(F) \subseteq \Phi^{+}_{\mathcal{A}}$$





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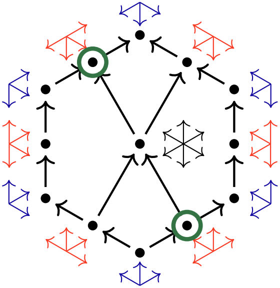
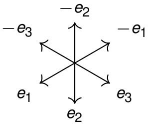




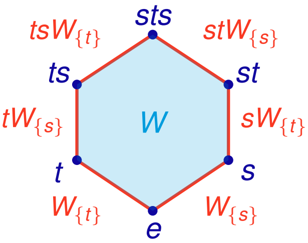
Root inversion sets

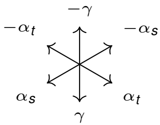
Definition (Root Inversion Set)

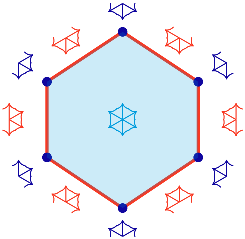
Let xW_I be a standard parabolic coset. The *root inversion set* is the set

$$\mathbf{R}(xW_I) := x(\Phi^- \cup \Phi_I^+)$$

Note that $N(x) = \mathbf{R}(xW_\varnothing) \cap \Phi^+$.



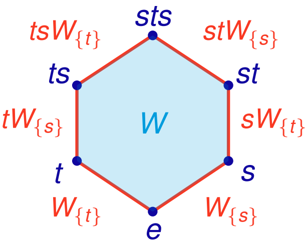


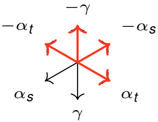


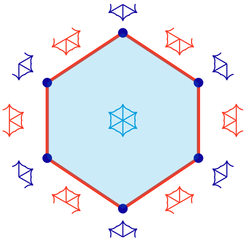
Root inversion sets

Example

$$\mathbf{R}(sW_{\{t\}}) = s(\Phi^- \cup \Phi^+_{\{t\}})$$
$$= s(\{-\alpha_s, -\alpha_t, -\gamma\} \cup \{\alpha_t\})$$
$$= \{\alpha_s, -\gamma, -\alpha_t, \gamma\}$$







Root inversion sets

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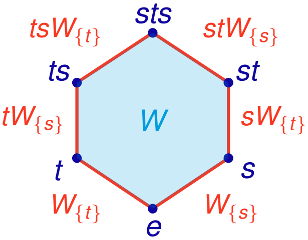
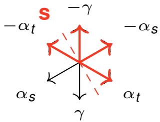
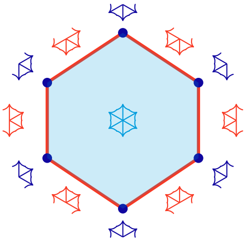


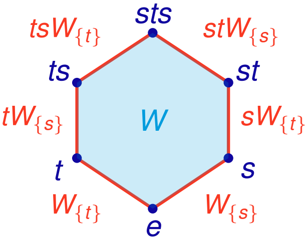


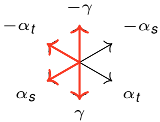


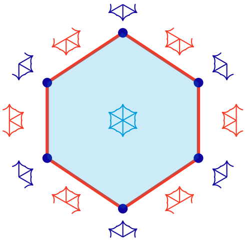
Root inversion sets

Example

$$\begin{aligned}\mathbf{R}(sW_{\{t\}}) &= s(\Phi^- \cup \Phi^+_{\{t\}}) \\ &= s(\{-\alpha_s, -\alpha_t, -\gamma\} \cup \{\alpha_t\}) \\ &= \{\alpha_s, -\gamma, -\alpha_t, \gamma\}\end{aligned}$$







Root inversion sets

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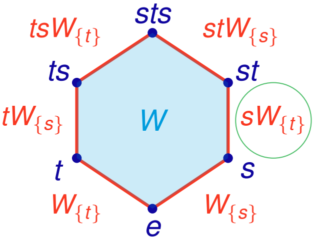
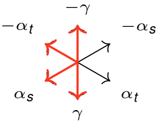
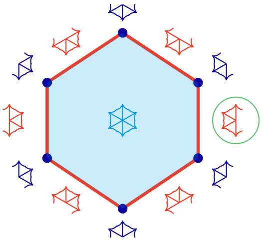




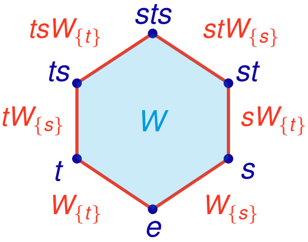


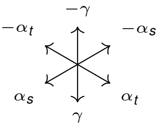
Root inversion sets

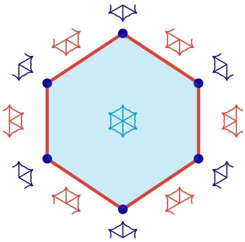
Proposition (D., Hohlweg, Pilaud '18)

Let xW_I be a standard parabolic coset of W. Then

$$\textbf{inner primal cone}\,(\mathbf{F}(xW_I)) = \mathrm{cone}\,(\mathbf{R}(xW_I))\,.$$







Equivalent definitions

Theorem (D., Hohlweg, McConville, Pilaud '19+)

For $F, G \in \mathscr{F}_{\mathcal{A}}$ the following are equivalent:

- $m_F \leq_{\mathrm{PR}} m_G$ *and* $M_F \leq_{\mathrm{PR}} M_G$ *in poset of regions* $\mathrm{PR}(\mathcal{A}, B)$.
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$$F = F_1 \lessdot_{\mathrm{F}} F_2 \lessdot_{\mathrm{F}} \cdots \lessdot_{\mathrm{F}} F_n = G$$

- $\mathbf{R}(F) \backslash \mathbf{R}(G) \subseteq \Phi^-_{\mathcal{A}}$ *and* $\mathbf{R}(G) \backslash \mathbf{R}(F) \subseteq \Phi^+_{\mathcal{A}}$.

Equivalent definitions

Theorem (D., Hohlweg, Pilaud '19)

The following conditions are equivalent for two standard parabolic cosets xW_I and yW_J in the Coxeter complex $\mathcal{P}_W$

- $x \leq_{\mathrm{R}} y$ *and* $xw_{\circ,I} \leq_{\mathrm{R}} yw_{\circ,J}$.
- $xW_I \leq_{\mathrm{COV}} yW_J$
- $\mathbf{R}(xW_I) \smallsetminus \mathbf{R}(yW_J) \subseteq \Phi^-$ *and* $\mathbf{R}(yW_J) \smallsetminus \mathbf{R}(xW_I) \subseteq \Phi^+$.

A super quick recap - Coxeter groups

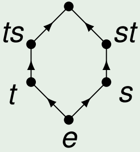
- (W, S) Coxeter system with
  $W := \langle s \in S \mid (s_i s_j)^{m_{i,j}} = e \text{ for } s_i, s_j \in S \rangle$.
- (right) weak order $\leq_R$ - $w \to ws$ and $\ell(w) < \ell(ws)$.

Example

Let Γ_{A_2} : 

.

$sts = w_\circ = tst$

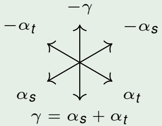


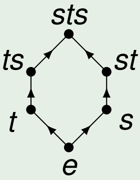
A super quick recap - Coxeter groups

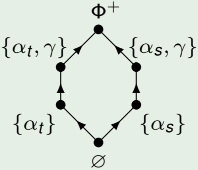
- Root system $\Phi = \{\alpha \in V \mid s_\alpha \in W\}$.
- (left) inversion set $\mathbf{N}(w) := \Phi^+ \cap w(\Phi^-)$.
- $w \leq_R u$ if and only if $\mathbf{N}(w) \subseteq \mathbf{N}(u)$.

Example

Let Γ_{A_2} : 

, with Φ given by the roots 

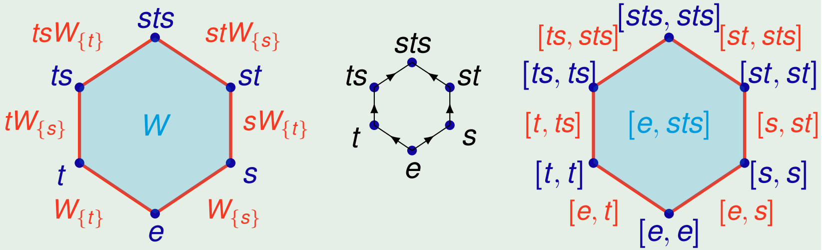




A super quick recap - Coxeter groups

- $W_I = \langle I \rangle$ for $I \subseteq S$.
- xW_I with $x \in W^I$ is a standard parabolic coset.
- Facial interval: $xW_I = [x, xw_{\circ,I}]$

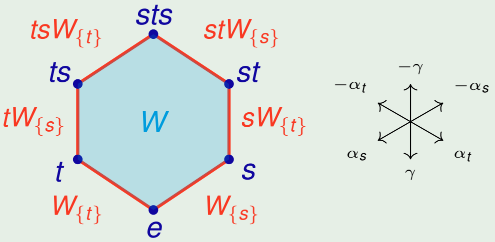
Example

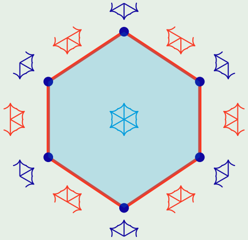


A super quick recap - Coxeter groups

- $\mathbf{R}(xW_I) = x\left(\Phi^- \cup \Phi_I^+\right)$

Example





A super quick recap - Coxeter groups - Facial weak order

- Cover relations / original definition:

$$(1) \qquad xW_I \lessdot_{\mathrm{COV}} xW_{I\cup\{s\}} \qquad \text{if } s \notin I \text{ and } x \in W^{I\cup\{s\}},$$

$$(2) \qquad xW_I \lessdot_{\mathrm{COV}} xw_{\circ,I}w_{\circ,I\smallsetminus\{s\}}W_{I\smallsetminus\{s\}} \qquad \text{if } s \in I,$$

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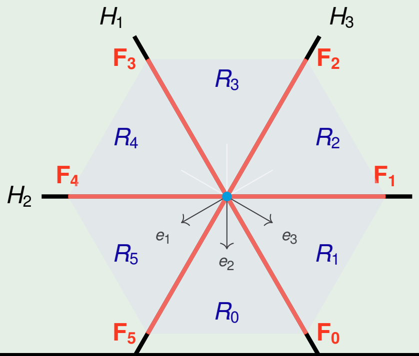
A super quick recap - Hyperplane arrangements

- $\mathcal{A} = \{H_1, \ldots, H_k\}$ is a (central, essential) arrangement.
- $\mathscr{R}_{\mathcal{A}}$ is the set of regions ($V \smallsetminus \mathcal{A}$)
- $\mathscr{F}_{\mathcal{A}}$ is the set of faces (intersections of region closures)

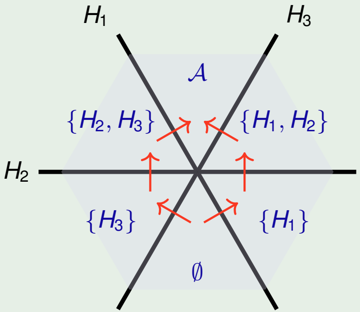
Example



A super quick recap - Hyperplane arrangements

- Poset of regions $\mathrm{PR}(\mathcal{A}, B)$
- Simplicial - every region has n bounding hyperplanes
- If $\mathcal{A}$ is simplicial then $\mathrm{PR}(\mathcal{A}, B)$ is a lattice.

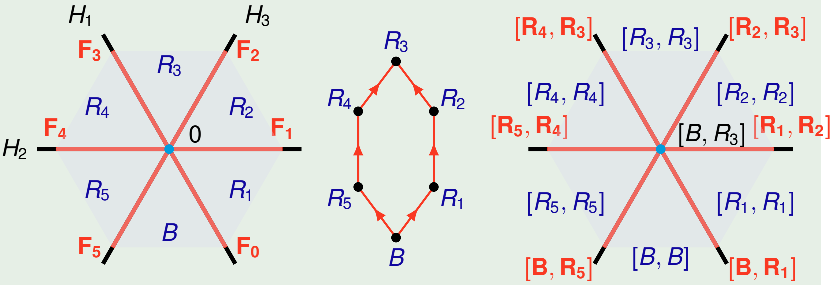
Example



A super quick recap - Hyperplane arrangements

- Facial interval of face F - $[m_F, M_F]$ in $\mathrm{PR}(\mathcal{A}, B)$.

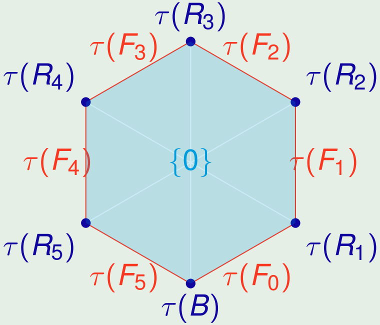
Example

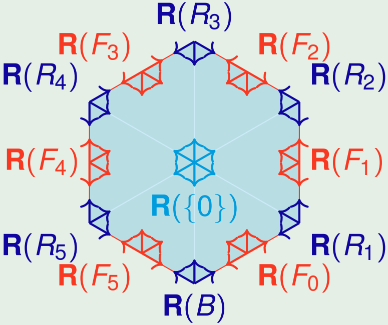


A super quick recap - Hyperplane arrangements

- Root inversion set:
  $\mathbf{R}(F) := \{e \in \Phi_{\mathcal{A}} \mid \langle x, e\rangle \leq 0 \text{ for some } x \in F\}$.

Example





A super quick recap - Facial weak order

Proposition (D., Hohlweg, McConville, Pilaud, '19+)

For $F, G \in \mathscr{F}_{\mathcal{A}}$ *if* $|\dim(F) - \dim(G)| = 1$ *and*

1. $F \subseteq G$, $M_F = M_G$, *or*
2. $G \subseteq F$, $m_F = m_G$.

then $F \lessdot_{\mathrm{F}} G$.

Theorem (D., Hohlweg, McConville, Pilaud '19+)

For $F, G \in \mathscr{F}_{\mathcal{A}}$ *the following are equivalent:*

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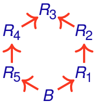
Equivalence for type A_2 Coxeter arrangement



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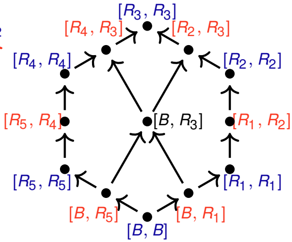

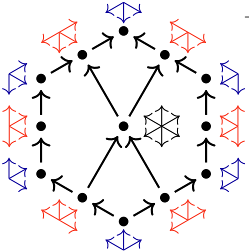

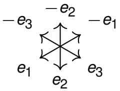

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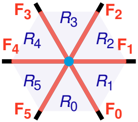

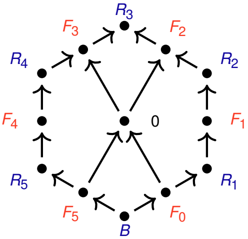

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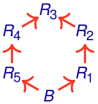

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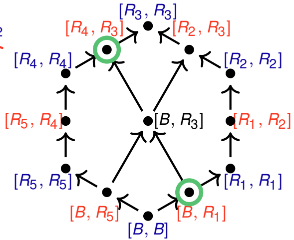

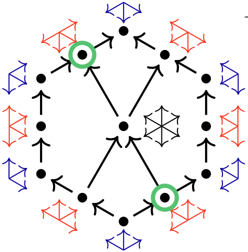

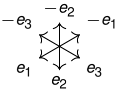

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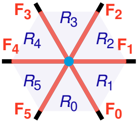

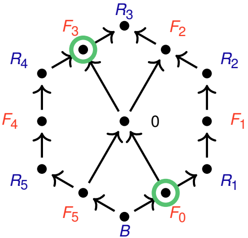

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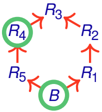

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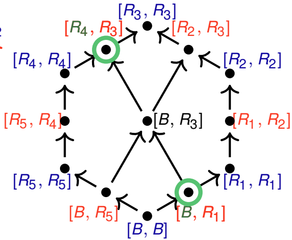

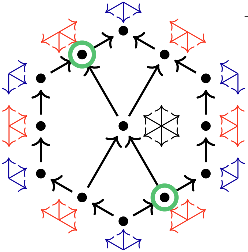

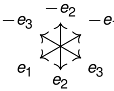

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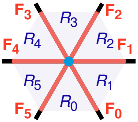

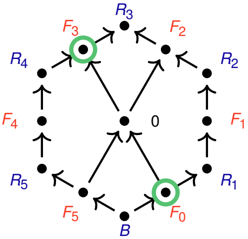

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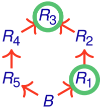

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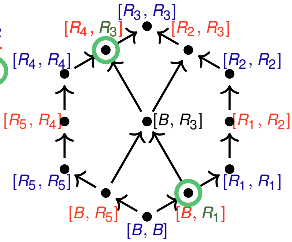

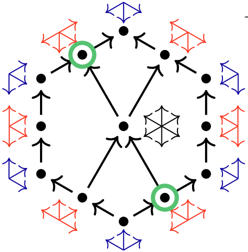

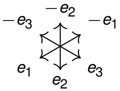

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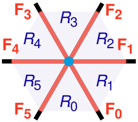

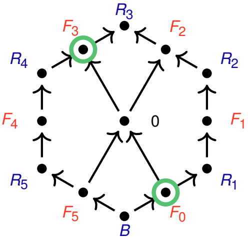

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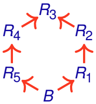

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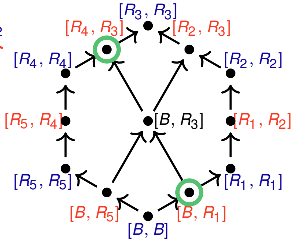

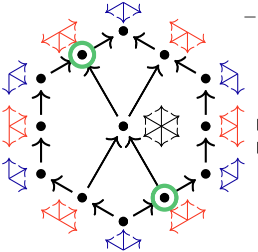

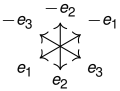

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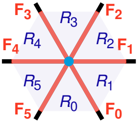

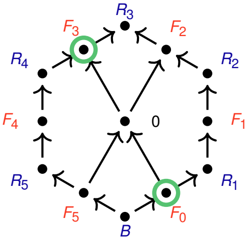

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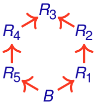

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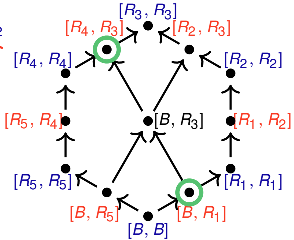

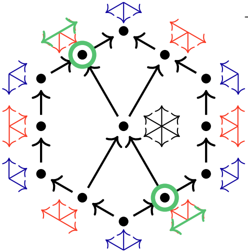

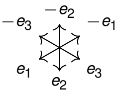

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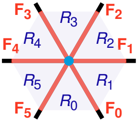

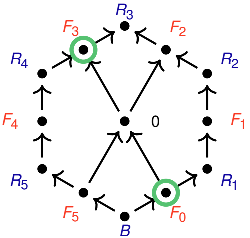

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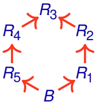

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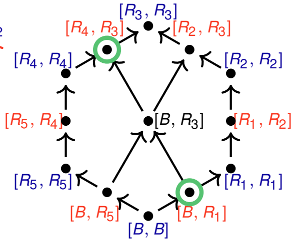
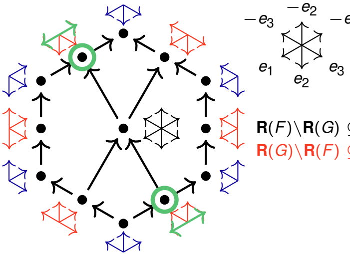
$M_F \leq_{PR} M_G$

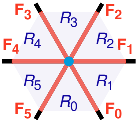


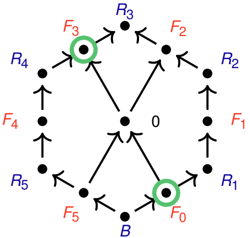


$\mathbf{R}(F) \setminus \mathbf{R}(G) \subseteq \Phi^-_{\mathcal{A}}$
$\mathbf{R}(G) \setminus \mathbf{R}(F) \subseteq \Phi^+_{\mathcal{A}}$

$|\dim F - \dim G| = 1$

- $F \subseteq G,\ M_F = M_G$, or
- $G \subseteq F,\ m_F = m_G$





Facial weak order lattice

Theorem (D., Hohlweg, Pilaud '19)

The facial weak order $(\mathcal{P}_W, \leq_F)$ is a lattice with the meet and join of two standard parabolic cosets xW_I and yW_J given by:

$$xW_I \wedge yW_J = z_\wedge W_{K_\wedge},$$
$$xW_I \vee yW_J = z_\vee W_{K_\vee}.$$

where,

$z_\wedge = x \wedge y$ *and* $K_\wedge = D_L(z_\wedge^{-1}(xw_{\circ,I} \wedge yw_{\circ,J}))$, *and*

$z_\vee = xw_{\circ,I} \vee yw_{\circ,J}$ *and* $K_\vee = D_L(z_\vee^{-1}(x \vee y))$

Corollary (D., Hohlweg, Pilaud '19)

The weak order is a sublattice of the facial weak order lattice.

Facial weak order lattice

Theorem (D., Hohlweg, McConville, Pilaud '19+)

Let $\mathcal{A}$ be an arrangement and fix a base region B. If the poset of regions $\mathrm{PR}(\mathcal{A}, B)$ is a lattice then the facial weak order $\mathrm{FW}(\mathcal{A}, B)$ is a lattice.

Corollary (D., Hohlweg, McConville, Pilaud '19+)

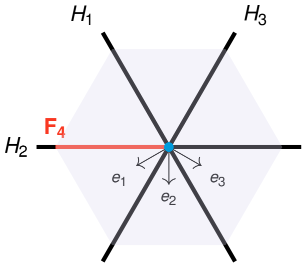
The lattice of regions is a sublattice of the facial weak order lattice when $\mathcal{A}$ is simplicial.

Covectors

- *covector* - a sign vector in $\{-, 0, +\}^{\mathcal{A}}$ with signs relative to hyperplanes.
- $\mathcal{L} \subseteq \{-, 0, +\}^{\mathcal{A}}$ - set of covectors

Example

$$F_4(H_1) = +;\ F_4(H_2) = 0;\ F_4(H_3) = - \qquad F_4 \leftrightarrow (+, 0, -)$$



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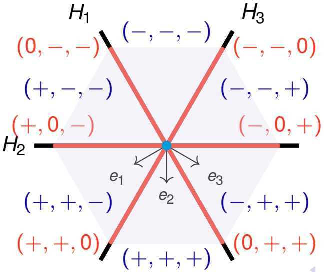
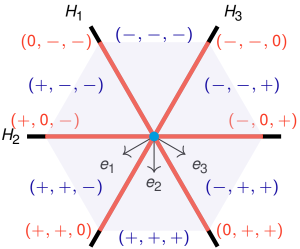


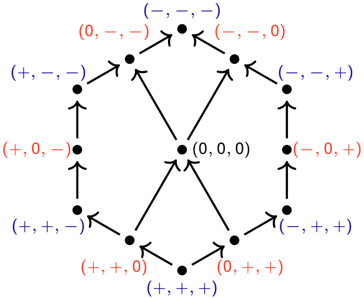
Covector definition

Definition

For $X, Y \in \mathcal{L}$:

$$X \leq_{\mathcal{L}} Y \iff X(H) \geq Y(H) \quad \forall H \text{ with } - < 0 < +$$





Equivalent definitions

Theorem (D., Hohlweg, McConville, Pilaud ’19+)

For $F, G \in \mathscr{F}_{\mathcal{A}}$ the following are equivalent:

- $F \leq_{\mathrm{F}} G$
- $F \leq_{\mathcal{L}} G$ *in terms of covectors* ($F(H) \geq G(H)\ \forall H \in \mathcal{A}$)

Covector operations

For $X, Y \in \mathcal{L} \subseteq \{-, 0, +\}^{\mathcal{A}}$

- *Composition:* $(X \circ Y)(H) = \begin{cases} Y(H) & \text{if } X(H) = 0 \\ X(H) & \text{otherwise} \end{cases}$
- *Reorientation:* $(X_{-Y})(H) = \begin{cases} -X(H) & \text{if } Y(H) = 0 \\ X(H) & \text{otherwise} \end{cases}$

⋆ For $F \in \mathscr{F}_{\mathcal{A}}$, $[m_F, M_F] = [F \circ B, F \circ -B]$

Example

Let $\mathcal{A} = \{H_1, H_2, H_3, H_4, H_5\}$.

$$X = (-, 0, +, +, 0) \qquad Y = (0, 0, -, 0, +)$$

Then

$$X \circ Y = (-, 0, +, +, +) \qquad X_{-Y} = (+, 0, +, -, 0)$$

Lattice proof - Joins

Proof uses two key components :

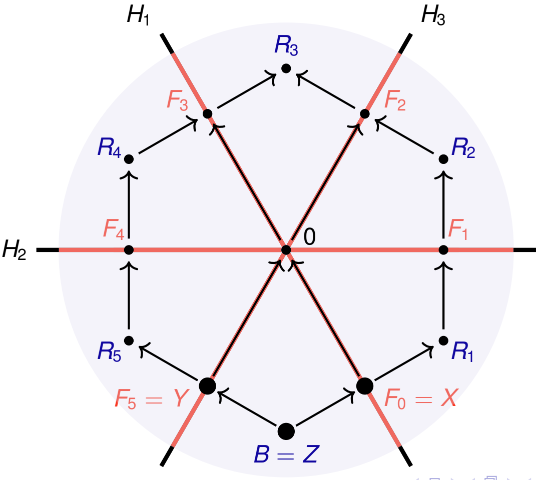
Lemma (Björner, Edelman, Ziegler '90)

1: If L is a finite, bounded poset such that $x \vee y$ exists whenever x and y both cover some $z \in L$, then L is a lattice.

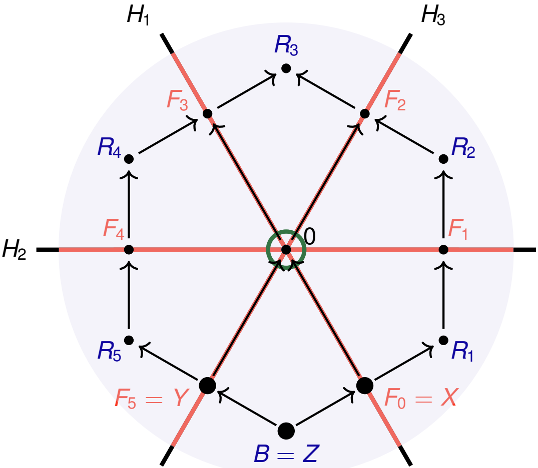
2: Cover relation: $Z \lessdot X$ iff $|\dim X - \dim Z| = 1$ and $Z \subseteq X$, $M_Z = M_X$ or $X \subseteq Z$, $m_Z = m_X$. Then $Z \lessdot X$ and $Z \lessdot Y$ gives three cases:

1. $X \cup Y \subseteq Z$, $m_X = m_Y = m_Z$ and $\dim X = \dim Y = \dim Z - 1$,
2. $Z \subseteq X \cap Y$, $M_X = M_Y = M_Z$ and $\dim X = \dim Y = \dim Z + 1$, and
3. $X \subseteq Z \subseteq Y$, $m_X = m_Z$, $M_Y = M_Z$ and $\dim X = \dim Z - 1 = \dim Y - 2$.

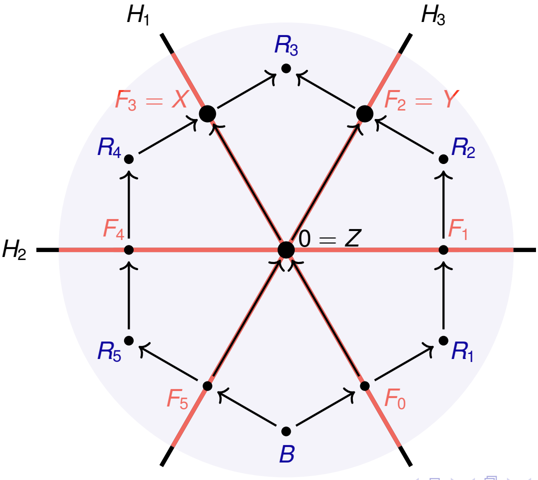
1. $X \cup Y \subseteq Z$, $m_X = m_Y = m_Z$ and $\dim X = \dim Y = \dim Z - 1$



1. $X \cup Y \subseteq Z$, $m_X = m_Y = m_Z$ and $\dim X = \dim Y = \dim Z - 1$



2. $Z \subseteq X \cap Y$, $M_X = M_Y = M_Z$ and $\dim X = \dim Y = \dim Z + 1$



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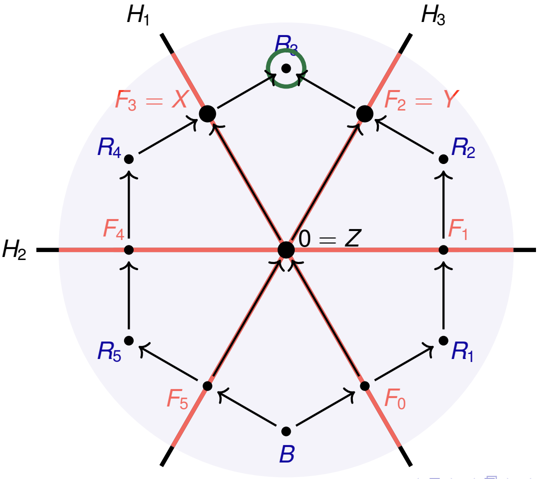
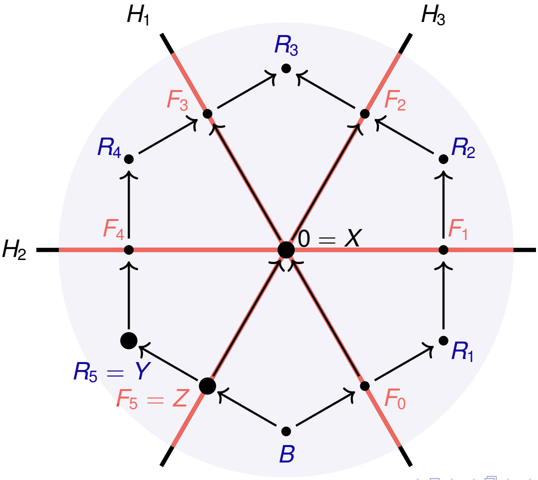


3. $X \subseteq Z \subseteq Y$, $m_X = m_Z$, $M_Y = M_Z$ and $\dim X = \dim Z - 1 = \dim Y - 2$



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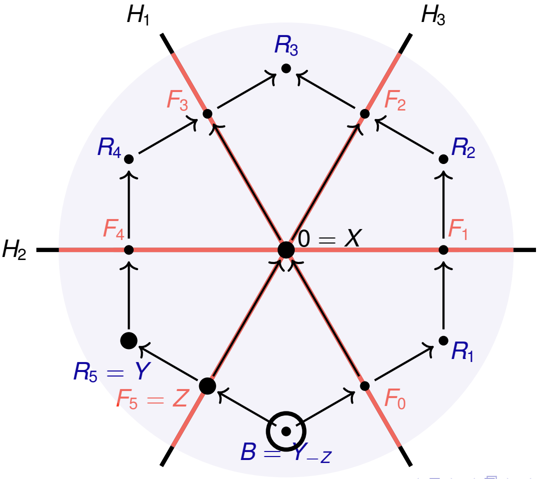

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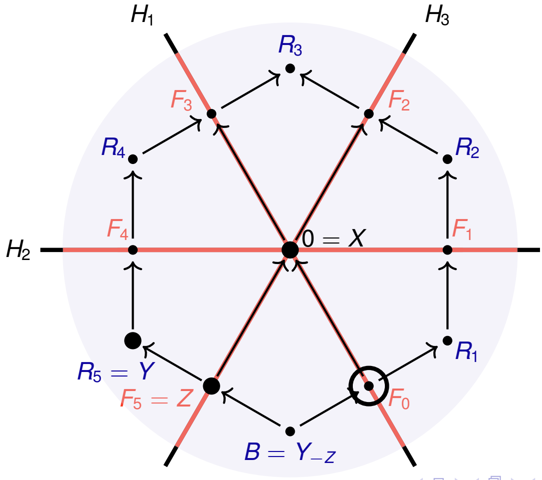
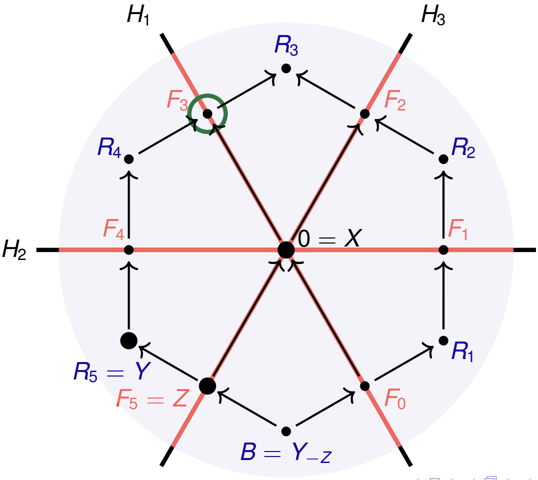
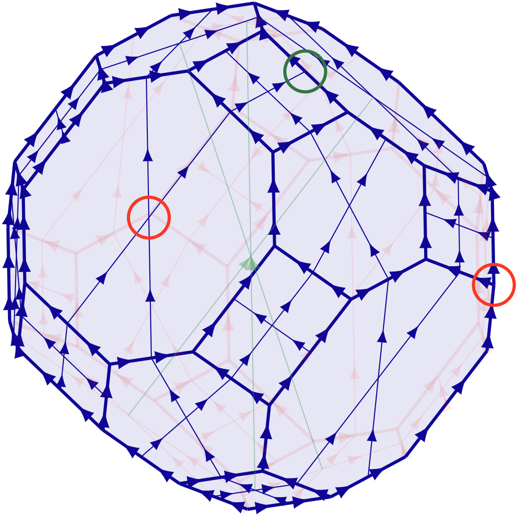


3. $X \subseteq Z \subseteq Y$, $m_X = m_Z$, $M_Y = M_Z$ and $\dim X = \dim Z - 1 = \dim Y - 2$



Example: B_3 Coxeter arrangement



Properties of the facial weak order

- The *dual* of a poset P is the poset P^{op} where $x \leq y$ in P iff $y \leq x$ in P^{op}. A poset is *self-dual* if $P \cong P^{op}$.
- A lattice is *semi-distributive* if $x \vee y = x \vee z$ implies $x \vee y = x \vee (y \wedge z)$ and similarly for the meets.

Theorem (D., Hohlweg, McConville, Pilaud '19+)

The facial weak order $\mathrm{FW}(\mathcal{A}, B)$ *is self-dual. If furthermore,* $\mathcal{A}$ *is simplicial,* $\mathrm{FW}(\mathcal{A}, B)$ *is a semi-distributive lattice.*

Join-irreducible elements

- An element is *join-irreducible* if and only if it covers exactly one element.

Proposition (D., Hohlweg, McConville, Pilaud '19+)

If $\mathcal{A}$ is simplicial and F a face with facial interval $[m_F, M_F]$. Then F is join-irreducible in $\mathrm{FW}(\mathcal{A}, B)$ if and only if M_F is join-irreducible in $\mathrm{PR}(\mathcal{A}, B)$ and $\mathrm{codim}(F) \in \{0, 1\}$

Möbius function

Recall that the Möbius function is given by:

$$\mu(x,y) = \begin{cases} 1 & \text{if } x = y \\ -\sum_{x \le z < y} \mu(x,z) & \text{if } x < y \\ 0 & \text{otherwise} \end{cases}$$

Proposition (D., Hohlweg, McConville, Pilaud '19+)

Let X and Y be faces such that $X \le Y$ and let $Z = X \cap Y$.

$$\mu(X,Y) = \begin{cases} (-1)^{\mathrm{rk}(X)+\mathrm{rk}(Y)} & \textit{if } X \le Z \le Y \textit{ and } Z = X_{-Z} \cap Y \\ 0 & \textit{otherwise} \end{cases}$$

Lattice Congruences

Definition

A *lattice congruence* is an equivalence relation $\equiv$ on a lattice $(L, \leq)$ such that for each $x_1 \equiv x_2$ and $y_1 \equiv y_2$ then

1. $x_1 \wedge y_1 \equiv x_2 \wedge y_2$, and
2. $x_1 \vee y_1 \equiv x_2 \vee y_2$.

Theorem (D., Hohlweg, Pilaud '19)

Given a lattice congruence $\equiv$ on $(W, \leq_R)$, the equivalence classes on $(\mathcal{P}_W, \leq_F)$ defined by

$$xW_I \,\overline{\equiv}\, yW_J \iff x \equiv y \text{ and } xw_{\circ,I} \equiv yw_{\circ,J}$$

give us a lattice congruence.

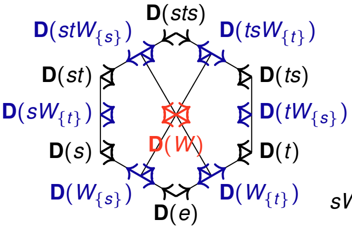
Facial Boolean Lattice

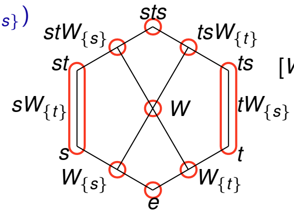
Corollary (D., Hohlweg, Pilaud '19)

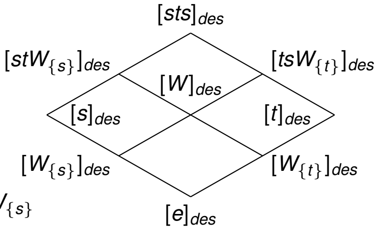
Let the (left) root descent set of a coset xW_I be the set of roots

$$\mathbf{D}(xW_I) := \mathbf{R}(xW_I) \cap \pm\Delta \ \subseteq \ \Phi.$$

Let $xW_I \equiv^{\mathrm{des}} yW_J$ if and only if $\mathbf{D}(xW_I) = \mathbf{D}(yW_J)$.



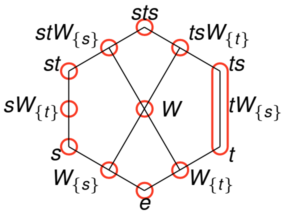


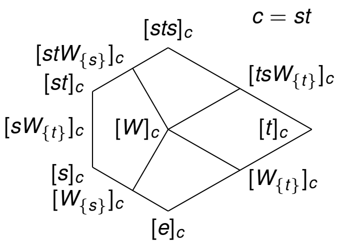


Facial Cambrian Lattice

Corollary (D., Hohlweg, Pilaud '19)

Let c be any Coxeter element of W. Let $\equiv^c$ be the c-Cambrian congruence (due to Reading [Cambrian Lattice, 2004]). Then let $xW_I \equiv^c yW_J \iff x \equiv^c y$ and $xw_{\circ,I} \equiv^c yw_{\circ,J}$.





Congruence normal

Definition

A lattice is *congruence normal* if it can be obtained from the 1-element lattice by a series of doublings of convex sets.

Example

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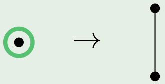

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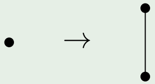

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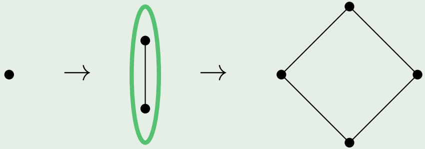

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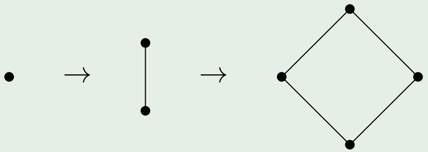

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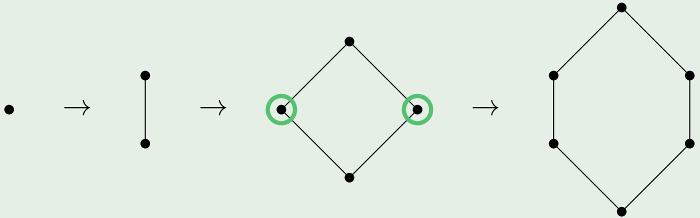


Congruence uniform

Let L be a finite lattice and $J(L)$ be the join-irreducibles.

- $Con(L)$ is the poset of lattice congruences partially ordered by refinement.
- L is *congruence uniform* if $J(Con(L)) \to J(L)$ is a bijection and similarly for meets.

Theorem (Day '94)

Let L be a finite lattice. The following are equivalent:

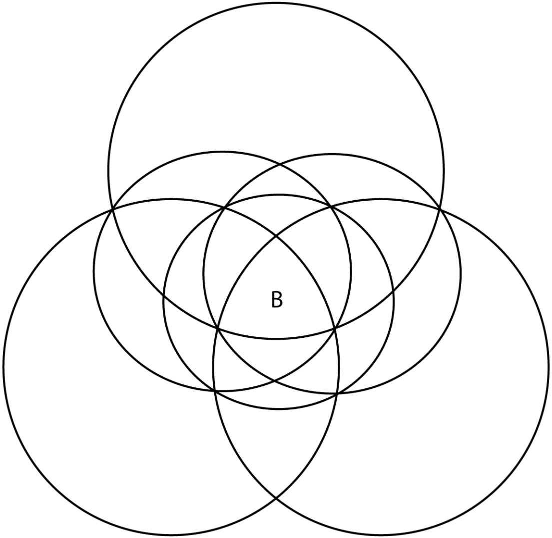
1. *L is congruence uniform*
2. *L is semi-destributive and congruence normal*
3. *L can be obtained from the* 1*-element lattice by a series of doublings of intervals.*

Congruence uniform

Theorem (Caspard, Conte de Poly-Barbut, Morvan '04)

The weak order is congruence uniform.

Shards



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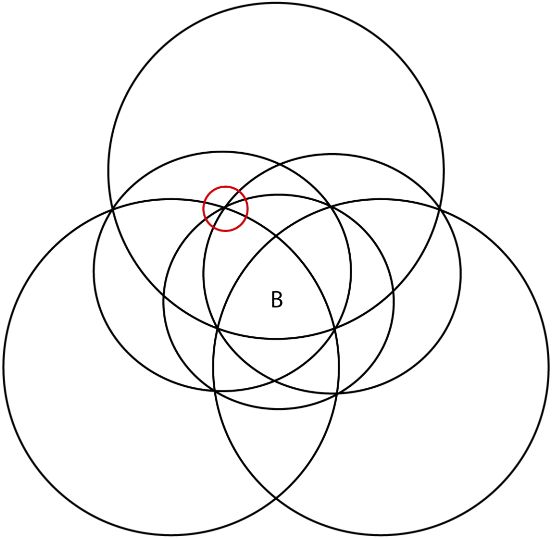

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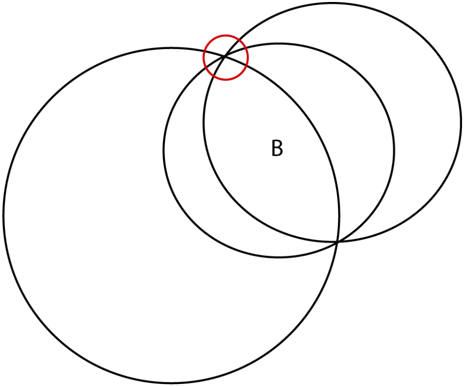

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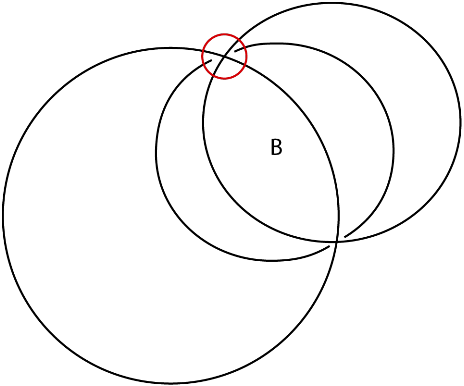

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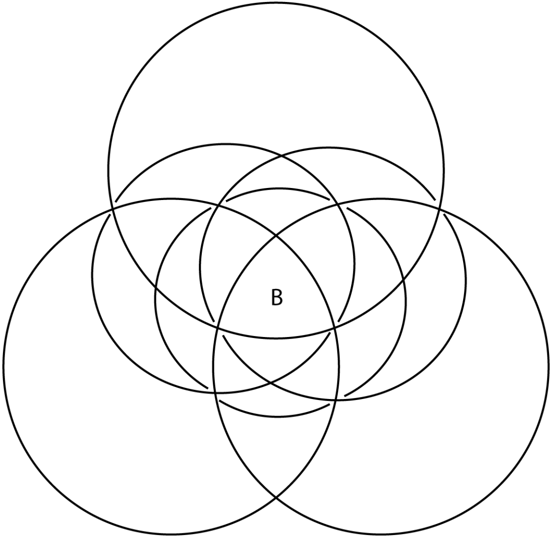
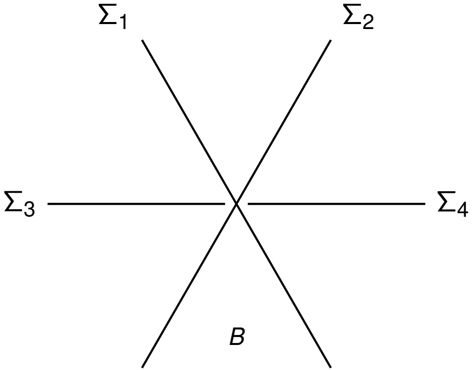


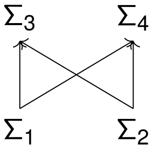
Shard intersection graph

Let $\mathsf{Sh}(\mathcal{A}, B)$ denote the set of shards.

Definition

For $\Sigma, \Sigma' \in \mathsf{Sh}(\mathcal{A}, B)$ let $\Sigma \to \Sigma'$ if and only if Σ "cuts" Σ'.



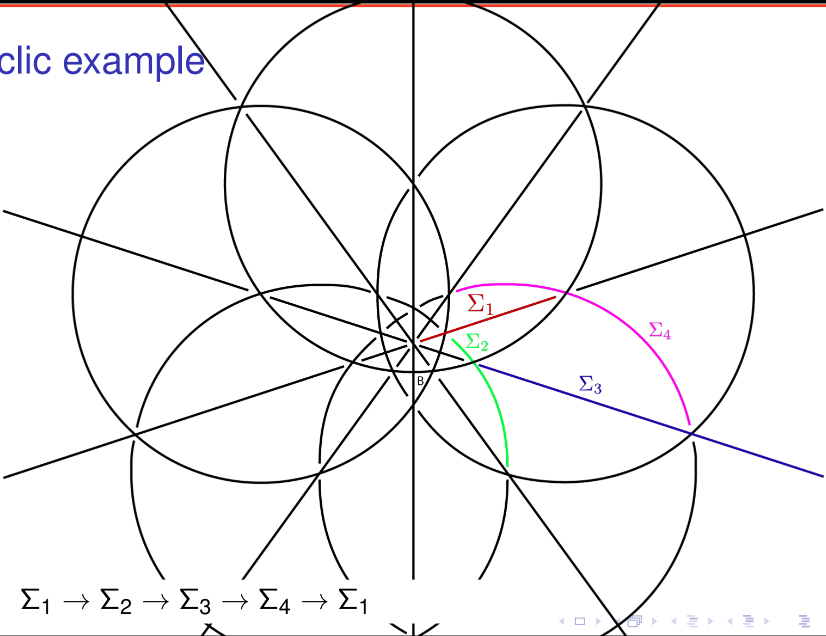


Congruence uniform and shard intersection graph

Theorem (Reading '04)

Let $\mathcal{A}$ be a simplicial arrangement. The lattice $\mathrm{PR}(\mathcal{A}, B)$ *is congruence uniform if and only if* $\mathrm{Sh}(\mathcal{A}, B)$ *is acyclic.*

Cyclic example



$\Sigma_1 \to \Sigma_2 \to \Sigma_3 \to \Sigma_4 \to \Sigma_1$

A nice conjecture

The *normal fan* of a polytope, is the collection of normal cones for every face.

Conjecture (Padrol, Pilaud, Ritter '20)

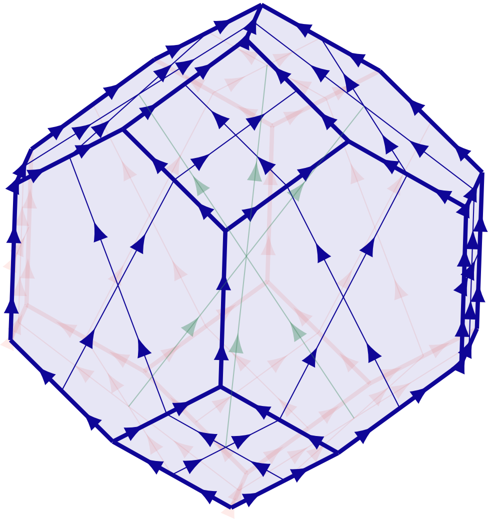
Let $\mathcal{A}$ be an arrangement whose zonotope has normal fan $\mathcal{F}$. Furthermore, suppose that $\mathrm{PR}(\mathcal{A}, B)$ is a congruence uniform lattice and $\equiv$ is any lattice congruence of $\mathrm{PR}(\mathcal{A}, B)$. Then the quotient fan $\mathcal{F}_{\equiv}$ is the normal fan of a polytope.

Conjecture (Padrol, Pilaud, Ritter '20)

Let $\mathcal{A}$ be an arrangement such that $\mathrm{PR}(\mathcal{A}, B)$ is a congruence uniform lattice. Then every shard admits a shard polytope.

Further Works

- Can we explicitly state the join/meet of two elements?
- When is the facial weak order congruence uniform?
- What happens when we look at shards?
- Can we generalize this to polytopes?



Thank you!