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ORDRE FAIBLE FACIAL, GROUPES DE COXETER ET MATROÏDES ORIENTÉS

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ARAM DERMENJIAN

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FACIAL WEAK ORDER, COXETER GROUPS AND ORIENTED MATROIDS

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ARAM DERMENJIAN

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On continue notre étude de la généralisation de l'ordre faible facial au contexte des arrangements d'hyperplans. On commence par donner les notions de base sur les arrangements d'hyperplans, les faces d'un arrangement et le poset des régions. De plus, on donne la définition d'un matroïde orienté et de ses covecteurs (une généralisation d'un arrangement d'hyperplans central et de ses faces). Ensuite, on fournit quatre caractérisations de l'ordre faible facial pour les arrangements d'hyperplans : comme un sous-poset induit du poset des intervalles du poset des régions, en donnant ses relations de couverture, en utilisant les covecteurs du matroïde orienté associé à l'arrangement et en utilisant la structure géométrique d'un zonotope associé à l'arrangement. On termine la section sur les arrangements d'hyperplans en utilisant ces caractérisations pour montrer que l'ordre faible facial sur les arrangements d'hyperplans simpliciaux est un treillis. Ceci est une extension d'un résultat bien connu de Björner, Edelman et Ziegler, établi en 1990 pour le poset des régions.

On conclut cette thèse en décrivant des problèmes ouverts et des directions pour l'avenir de cette recherche.

Mots clés: ordre faible, groupes de Coxeter, arrangements d'hyperplans, poset de regions, matroïdes orientés, poset de topes, permutoèdre, zonotopes, quotients de treillis, associaèdre

ABSTRACT

We investigate the *facial weak order*, a poset structure that was first introduced by Krob, Latapy, Novelli, Phan and Schwer at FPSAC 2001 on the symmetric groups. The facial weak order extends the weak order on a finite Coxeter group W to the set of all faces of the permutahedron of W. We first give the necessary background material on Coxeter groups, the weak order and permutahedra before defining the facial weak order on Coxeter groups. We then provide three characterizations of this poset: the original one in terms of cover relations (extended from the type A case to all Coxeter groups by Palacios and Ronco in 2006), the geometric one that generalizes the notion of inversion sets and the combinatorial one as an induced subposet of the poset of intervals of the weak order. We end the Coxeter group part of this thesis by using these characterizations to show that the facial weak order on Coxeter groups is in fact a lattice, extending a well-known result of Björner in 1984 for the classical weak order.

We continue our study by generalizing the facial weak order to the context of hyperplane arrangements. We begin with the necessary background on hyperplane arrangements, faces of an arrangement and the poset of regions in addition to background on oriented matroids and their covectors (a generalization of central hyperplane arrangements and their faces). We then provide four characterizations of the facial weak order for hyperplane arrangements: as an induced subposet of the poset of intervals of the poset of regions, by describing their cover relations, using covectors (from its associated oriented matroids) and using the geometric structure of the zonotope associated to the arrangement. We end the hyperplane arrangement part of this thesis by using these characterizations to show that the facial weak order on simplicial hyperplane arrangements is in fact a lattice, extending a well-known result of Björner, Edelman and Ziegler in 1990 for the poset of regions.

We conclude our thesis by describing open problems and further directions of research.

Keywords: weak order, Coxeter groups, hyperplane arrangements, poset of regions, oriented matroids, tope poset, permutahedra, zonotopes, lattice quotients, associahedra

INTRODUCTION

H.S.M. Coxeter and his groups

Suppose you are standing in front of a mirror, admiring your reflection. After a few hours of gazing at your perfection, you look past your reflection to notice another mirror directly behind you. A second mirror that makes your own reflection reflect, creating a kaleidoscope of never-ending reflections of your awesome self. Now suppose you decide to shake things up a bit and you move the mirror behind you so that it is next to the mirror in front of you. As you are placing this second mirror, you notice that the number of times you see yourself changes depending on the angle between the mirrors! As you decrease the angle between the mirrors, more and more reflections of your gorgeous self appear. This makes you wonder: what decides these reflections? Is there a way to know how many reflections of yourself are going to appear if we know the angle between the mirrors? Is there a nice way to generate and represent these reflections?

These were some of the first questions posed and answered by the Canadian-British mathematician Harold Scott MacDonald Coxeter (H.S.M. Coxeter or Donald Coxeter for short). Coxeter was known for walking around Cambridge (where he did his Ph.D.) with mirrors so that he could show their amazing properties to anyone and everyone that would listen.[1] The reflections in these mirrors form what is called a discrete reflection group. In 1934, while at Princeton, Coxeter showed

[1] For a history on Coxeter and his life, the reader is referred to the amazing book "King of Infinite Space: Donald Coxeter, the Man Who Saved Geometry" by Roberts (Roberts, 2006).

that all finite discrete reflection groups can be represented by certain abstract groups (see (Coxeter, 1934)). Then in 1935 he showed that these abstract groups represent all finite discrete reflection groups in (Coxeter, 1935). Tits in (Tits, 1961) called these abstract groups “groupes de Coxeter” (or “Coxeter groups” in English) and, from then on, in honour of Coxeter, the term Coxeter group has stuck around.

Due to their relationship with mirrors, Coxeter groups encode the symmetries of regular polyhedra. The different polyhedra are the different types of Coxeter groups. The type A Coxeter groups represent the symmetries of such objects as the triangle and the tetrahedron.[2] The type B Coxeter groups represent the symmetries of such objects as the square, the cube, and the octahedron. There are also the types D, E, F, H and I Coxeter groups, with I representing the symmetries of regular polygons.[3] The two other 3-dimensional regular polytopes are found in the type H_3 Coxeter group. The link between finite reflection groups and finite Coxeter groups are discussed in more precise mathematical terms from § 1.1 to § 1.3.

A. Björner names an order

Looking back at the two mirrors in front of you, you realize that there are many different ways to order your own reflections. You could order them in a clockwise order, a counter-clockwise order, from those reflections closest to you to the ones that are further away, or the reverse, and potentially in other weird ways (such as zigzags). Imagine what were to happen if you had a third mirror on top of you producing even more reflections! The way you could order these reflections is very

[2] The type A Coxeter groups are also commonly referred to as symmetric groups.

[3] The type I Coxeter groups are also commonly referred to as dihedral groups.

useful as it gives information on the structure of the group itself.

One of the most well-studied orders in Coxeter groups is called the weak order. The weak order is nothing more than ordering the reflections starting from the ones closest to yourself (yourself included!) and going "further away" from you. Think of it as a walk starting from yourself and into the mirror-worlds where you are only ever allowed to walk away from where you started. This order was first introduced in 1963 by Guilbaud and Rosenstiehl in (Guilbaud & Rosenstiehl, 1963) where it was created to help understand different ways to choose candidates from a ballot.

In their construction, Guilbaud and Rosenstiehl introduced the order only for the type A Coxeter groups. The name "weak order" and the generalization to all Coxeter groups would appear in 1984 by Björner in his article (Björner, 1984). This term was chosen as it was a weaker version of the strong order, also known as the Bruhat order. In this same article, Björner stated that, in finite cases, the weak order gives you a lattice, which for us translates to the fact that if two of your reflections wanted to meet each other, they can do so by either both of them walking strictly away from you or strictly toward you, going from one mirror world to the next, until they meet at some unique spot. In mathematical terms, a lattice can be thought of as an order in which every two elements has a least upper bound and a greatest lower bound. It is easy to imagine ways of placing your original mirrors so that there is no "furthest" reflection (such as if you are completely surrounded by mirrors or if you put the mirrors back in their starting positions: one in front of you and one directly behind you, so that they were parallel). In this case, Björner stated that the weak order is a meet-semilattice, which means that reflections can meet by walking strictly towards you, but not necessarily by walking strictly away from you (or only having a greatest lower bound but not necessarily a least upper bound). Topics related to the weak order and the weak order itself are discussed from sections § 1.4 to § 1.6.

Facial weak order

Another way to look at your reflections is to connect them all and construct a polygon out of them. This polygon is called a permutahedron whose origins began in (Schoute, 1911), but was first named in (Guilbaud & Rosenstiehl, 1963). In higher dimensions, the permutahedron has many smaller faces such as vertices, edges, polygons, etc. If we stand on one of the vertices of the permutahedron, then we can think of the weak order as nothing more than us walking along the edges from vertex to vertex where the vertices are ordered from closest to our starting point to furthest from our starting point. The permutahedron and various perspectives on it are discussed in the sections from § 1.7 to § 1.8.

Imagine now that we allow ourselves to walk not just on edges, but to also take shortcuts through any of the faces. This is the idea of the facial weak order. The facial weak order was first introduced by Krob, Latapy, Novelli, Phan and Schwer in (Krob et al., 2001) for the type A Coxeter groups at the *Formal Power Series and Algebraic Combinatorics* conference in 2001. They showed that the facial weak order for the type A Coxeter groups is a lattice; in other words, every two faces has a least upper bound and a greatest lower bound. As normally happens, the facial weak order was then generalized to all Coxeter groups. This generalization was done in 2006 by Palacios and Ronco in (Palacios & Ronco, 2006) where they gave a definition of the facial weak order on arbitrary Coxeter groups, but gave no indication on whether the order was a lattice or not. We will prove in Chapter 2 that the facial weak order does in fact produce a lattice for arbitrary finite Coxeter groups.

Hyperplane Arrangements

One thing that mathematicians love to do is to generalize results. After publishing (Dermenjian et al., 2018) we decided to generalize our results from Coxeter groups into hyperplane arrangements. This generalization is particularly interesting because it gives us a new way to view the weak order: through mirror hopping!

Enlarging each mirror in every direction forever, we obtain what is known as a hyperplane, a space that cuts the entire universe in two. If you have just one mirror in front of you, you can think of this mirror as the hyperplane, separating everything on your side of the mirror from everything on your reflection's side. By considering mirrors as things that cut everything in two, we observe that we are creating regions where each of your reflections live. Then you could visit your reflection by "hopping" through a hyperplane/mirror, just like Alice through her looking glass (Carroll, 1871). These mirrors form what is known as a hyperplane arrangement and the areas that your reflections live, regions.

Hyperplane arrangements have a rich history dating back to ancient times, ever since civilization started sharing things. In fact, you probably use hyperplane arrangements in your daily life! (And that is not even including mirrors!) Imagine, for instance, you are throwing a party and invited n people, but only have 1 cake (or block of cheese if you would prefer). Each time you cut the cake, you are introducing a hyperplane, cutting the entire universe (or in our case, the cake) in two. An age-old question then asks: if you make n number of cuts, how many slices will you have? This question was not formalized until 1826 with Steiner's first publication (Steiner, 1826) on laws of cutting 2-dimensional and 3-dimensional Euclidean spaces. Although Steiner was the first to publish according to Grünbaum (see the brief history in (Grünbaum, 2003, Chapter 18)), Steiner mentions in (Steiner, 1826) that many geometrical textbooks had started looking at hyperplane arrangements at the time of his publication.

Going back to our mirrors and the hyperplane arrangement we get from taking a finite number of mirrors, remember that we said a region is a place that each of your reflections live. If you start on any region and then do a sequence of mirror hoppings you can define an order on the regions much like we did in the previous sections for Coxeter groups. You can order your regions from the regions that are closest to you all the way to the regions that are furthest from you. This order, which gives the poset of regions, is the hyperplane arrangement equivalent to the weak order for Coxeter groups. Although hyperplane arrangements are old, this partial order was first given by Edelman in (Edelman, 1984). Only six years after, it was shown to be a lattice for large families of hyperplane arrangements by Björner, Edelman and Ziegler in (Björner et al., 1990). Unlike Coxeter groups, there are hyperplane arrangements which do not produce lattices for the poset of regions if you start from a "bad" region. An example of this is given when we discuss the poset of regions in § 3.4.

At this point you might be wondering what is the difference between hyperplane arrangements and Coxeter groups because both are using mirrors. The only difference lies in the angles between the mirrors! Since Coxeter groups come from finite reflection groups, the angles between the mirrors are forced[4] to be a fraction of the form $\frac{\pi}{n}$ where $n \geq 2$. In hyperplane arrangements, we remove this restriction, thus allowing for a much more general positioning of hyperplanes. Therefore, hyperplane arrangements are nothing more than a generalization of Coxeter groups

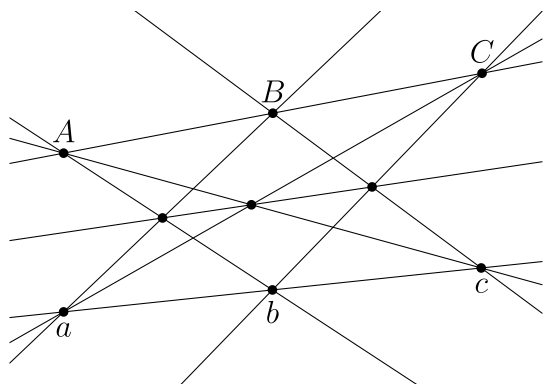
[4]It is slightly more complicated than this. Once you start choosing angles between some mirrors, by the nature of finite reflection groups, you no longer have a choice of angles between other mirrors. This is given in much more (precise) detail in Chapter 1.

and they are discussed in further details from sections § 3.1 to § 3.5.

Oriented Matroids

But why stop at hyperplane arrangements? Continuing down this journey we can generalize our results one step further to oriented matroids. Oriented matroids are a generalization of central hyperplane arrangements: hyperplane arrangements where every hyperplane contains some point in common (or in mirror land, a set of mirrors that all meet at some point). According to the historical remarks in (Björner et al., 1999) oriented matroids have a very rich history. The first few papers that lead towards general oriented matroids started with (Minty, 1966). Minty's paper was likely an inspiration to (Fulkerson, 1968) and (Rockafellar, 1969), the latter being one of the first to propose the axiomatization of general oriented matroids. The credit for the origins of oriented matroids is usually attributed to Bland, Folkman, Las Vergnas and Lawrence who worked on oriented matroids in the late 1960s and early 1970s. Folkman, in particular, started working on oriented matroids in 1967, but never got to finish his work as he died before publishing. His work was then advanced by Lawrence in 1975 in his doctoral thesis. Bland and Las Vergnas had also independently stumbled upon oriented matroids (Bland through linear programming and Las Vergnas through oriented graphs) in 1974. Each of these four made substantial contributions to the theory and much of the foundation for oriented matroids was laid out by these four people.

For us, the idea of oriented matroids is to generalize central hyperplane arrangements using some of the most crucial properties of hyperplanes: that they split the space into two halves and contain the origin. Thus we can think of oriented matroids as objects whose elements, called covectors, store information on "sidedness". In terms of mirrors, this idea of "sidedness" basically states whether an object is on your side of a mirror or on the other side. By removing hyperplanes from the equation and only looking at the concept of sidedeness, oriented matroids do not necessarily follow geometrical rules in normal (Euclidean) space. In fact, one of the most famous oriented matroids which breaks a geometric rule is the one that goes “against” Pappus’ hexagon theorem. Pappus’ hexagon theorem[5] states that if there is a line with the points A, B and C on it and another line with the points a, b and c on it, then the intersection points of the lines Ab and Ba, Ac and Ca, and Bc and Cb all lie on another straight line.



An example of the oriented matroid that breaks this theorem can be seen in Figure 3.10. It turns out that the oriented matroids that do not break geometric rules are exactly the oriented matroids which arise from hyperplane arrangements. We study oriented matroids in more detail from sections § 3.6 to § 3.8.

It turns out we can generalize the facial weak order of Coxeter groups to a facial weak order on (the covectors of) oriented matroids. In Chapter 4 we introduce this generalization and give multiple (equivalent) definitions of the facial weak order on the faces of a hyperplane arrangement. Recall that the facial weak order was an extension of the weak order to the faces of a Coxeter group (where we went from walking vertex to vertex to walking face to face on the permutahedron). In the

[5]See (Coxeter, 1989) for the exact phrasing on Pappus’ hexagon theorem.

case of hyperplane arrangements, the facial weak order is an extension of the poset of regions to the faces of the hyperplane arrangement. In (Björner et al., 1990) it was shown by Björner, Edelman and Ziegler that the poset of regions is a lattice for a special family of hyperplane arrangements called simplicial arrangements. Since the poset of regions is a lattice for simplicial arrangements, it would be reasonable to conjecture that the facial weak order would also be a lattice for simplicial arrangements. In Chapter 4 we show that this conjecture is correct and that the facial weak order is a lattice for simplicial arrangements. We then end this thesis by leaving the reader with a kaleidoscope of open problems and questions that we have for the facial weak order and hope to solve in the coming years.

CHAPTER I

COXETER GROUPS

The aim of this chapter is to prepare the reader for our first article (Dermenjian et al., 2018), which we present in Chapter 2. We start with an introduction to finite reflection groups and how they are related to finite Coxeter groups in § 1.1. In § 1.2 we give a formal definition of Coxeter groups. Focusing on the geometric aspect of Coxeter groups, we present in § 1.3 a geometric representation of a Coxeter group. The combinatorial aspect of Coxeter groups is then discussed by surveying the notions of reduced words and lengths of elements in § 1.4. In § 1.5 we review the notions of root systems and inversion sets that provide a geometrical tool to study reduced words and lengths of elements. We then describe a poset called the weak order in § 1.6 that is poset isomorphic to the inversion sets ordered by inclusion. The weak order is the order that we extend to the facial weak order in Chapter 2. Continuing this survey, we present the permutahedron in § 1.7, a polytope that encodes the structure of a finite Coxeter group. Finally, in § 1.8, we cover the notions of parabolic subgroups and their cosets and their relation with the permutahedron. This chapter contains no proofs. For a more thorough background on the topics covered in this chapter and for the proofs, the reader is invited to consult the book "Reflection groups and Coxeter groups" (Humphreys, 1990), the book "Combinatorics of Coxeter groups" (Björner & Brenti, 2005) and the book "Lie groups and Lie algebras: Chapters 4–6" (Bourbaki, 1968).

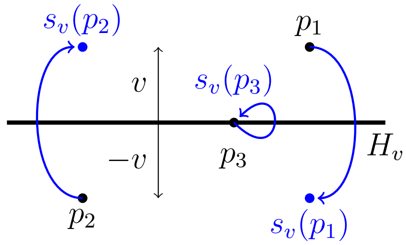


Figure 1.1: A vector v with the reflection s_v on the points p_1, p_2 and p_3 and the reflecting hyperplane H_v associated to v.

1.1 Finite reflection groups

Let $(V, \langle \cdot, \cdot \rangle)$ be an n-dimensional Euclidean space with positive definite symmetric bilinear form $\langle \cdot, \cdot \rangle$. For any vector $v \in V \setminus \{0\}$, let the linear operator s_v denote the *reflection* which interchanges v with $-v$ and fixes pointwise the hyperplane H_v orthogonal to v (see Figure 1.1 for an example). There is a simple formula for the reflection s_v given by the following equation:

$$s_v(u) = u - \frac{\langle u, v \rangle}{\langle v, v \rangle} v.$$

Notice that $s_v(v) = -v$ and $s_v(u) = u$ for $u \in H_v$ since $\langle u, v \rangle = 0$.

A *(finite) reflection group* is a finite group W generated by a set of reflections in the orthogonal group $O(n)$. The classical examples of finite reflection groups in the cases of dimension of V equal to 2 or 3 are the symmetry groups of the ordinary regular polygons and regular polyhedra. We give as an example the symmetry group of a (regular) equilateral triangle next.

Example 1.1.1 Consider an equilateral triangle and label its vertices by 1, 2 and 3 (see Figure 1.2). There are two reflections that will generate all possible labellings of the triangle. These two reflections are given in Figure 1.2 on the first triangle where the two dashed lines represent the hyperplanes associated to the two reflections. Reflecting over these two lines keeps the triangle invariant but changes the labels. For example, reflecting the first labelled triangle over the vertical line sends the first labelled triangle to the third one in Figure 1.2. Applying all possible compositions of these two reflections one gets only six possible labellings of the equilateral triangle (as seen in Figure 1.2). In fact, the group generated by these two reflections is a finite reflection group of order 6, which will later be known as the Coxeter group of type A_2. This group is also known as the symmetric group S_3.

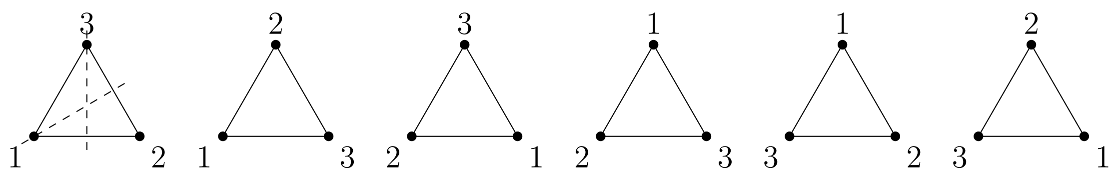


Figure 1.2: The symmetries of a triangle.

In 1934 Coxeter showed that finite reflection groups can be described using a particular set of reflections, called simple reflections, together with a certain set of relations.

Theorem 1.1.2 *(Coxeter, 1934, Theorem 8) Every finite reflection group W has a presentation by generators and relations of the form*

$$W = \langle S \mid (st)^{m_{s,t}} = e, \ s, t \in S \rangle ,$$

where S is a set of involutions, $m_{s,t} \in \mathbb{N}_{\geq 2}$ if $s \neq t$ and e is the identity element of W.

This presentation of finite reflection groups is named in honour of Coxeter and we study this presentation in the following section.

1.2 Coxeter groups

A *Coxeter group* is a group generated by a finite set S of *simple reflections* with the following presentation:

$$W = \langle S \mid \ (st)^{m_{s,t}} = e,\ s,t \in S \rangle$$

where $m_{s,t} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ when $s \neq t$ and $m_{s,s} = 1$. Here e denotes the identity of W and $m_{s,t} = \infty$ means that we impose no condition of the form $(st)^m = e$.

The term "Coxeter group" is ambiguous; it is necessary to specify the set of simple reflections generating the group. This is due to the fact that a group W could be generated by various sets of simple reflections in the sense of the definition above. For example, the dihedral group $\mathcal{D}_6$ of order 12 can be represented in either of the two following presentations:

$$\Big\langle \{a,b\} \mid \ a^2 = b^2 = (ab)^6 = e \Big\rangle$$

and

$$\Big\langle \{x,y,z\} \mid \ x^2 = y^2 = z^2 = (xy)^3 = (xz)^2 = (yz)^2 = e \Big\rangle .$$

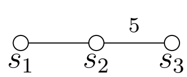
For this reason we say that a pair (W,S) is a *Coxeter system*, allowing us to unambiguously know which set of simple reflections generate a Coxeter group.

When a Coxeter group W is finite it is called a *finite Coxeter group*. In fact, finite Coxeter groups and finite reflection groups turn out to be the same, see for instance (Humphreys, 1990, Theorem 6.4).

Theorem 1.2.1 *A Coxeter group is finite if and only if it is a finite reflection group.*

Rather than explicitly writing the presentation of a Coxeter group each time, it is convenient to encode it combinatorially into a graph. Given a Coxeter system (W, S), the *Coxeter graph* (or *Coxeter diagram*) is the partially (edge-)labelled graph (S, V) where the vertex set is the set S of simple reflections and a pair of distinct simple reflections s and t share an edge if $m_{s,t} \geq 3$ which is labelled with $m_{s,t}$ if $m_{s,t} \geq 4$ or ∞. Examples of Coxeter graphs can be found in Figure 1.3 where the label on the left of each graph is known as the *type* of the Coxeter group. For a Coxeter system (W, S), the Coxeter group W is said to be *irreducible* if its Coxeter graph is connected; if its Coxeter graph is not connected, then W is said to be *reducible* and is the direct product of irreducible Coxeter groups (the connected components).

Example 1.2.2 As an example we use the type H_3 Coxeter group. From Figure 1.3 we can observe that the type H_3 Coxeter group is given by the Coxeter graph:



From the Coxeter graph we deduce that $S = \{s_1, s_2, s_3\}$. For the relations between the simple reflections we use the edges. As there is an unlabelled edge between the first two nodes we have $(s_1s_2)^3 = e$. As there is an edge labelled 5 between the second two nodes we have $(s_2s_3)^5 = e$. Finally, since the first and last nodes do not share an edge, these two elements commute, $(s_1s_3)^2 = e$. All of this is given in the presentation for W:

$$W = \left\langle \{s_1, s_2, s_3\} \mid \ (s_1)^2 = (s_2)^2 = (s_3)^2 = (s_1s_2)^3 = (s_1s_3)^2 = (s_2s_3)^5 = e \right\rangle.$$

As can be observed, using Coxeter graphs and types makes referring to Coxeter groups significantly easier.

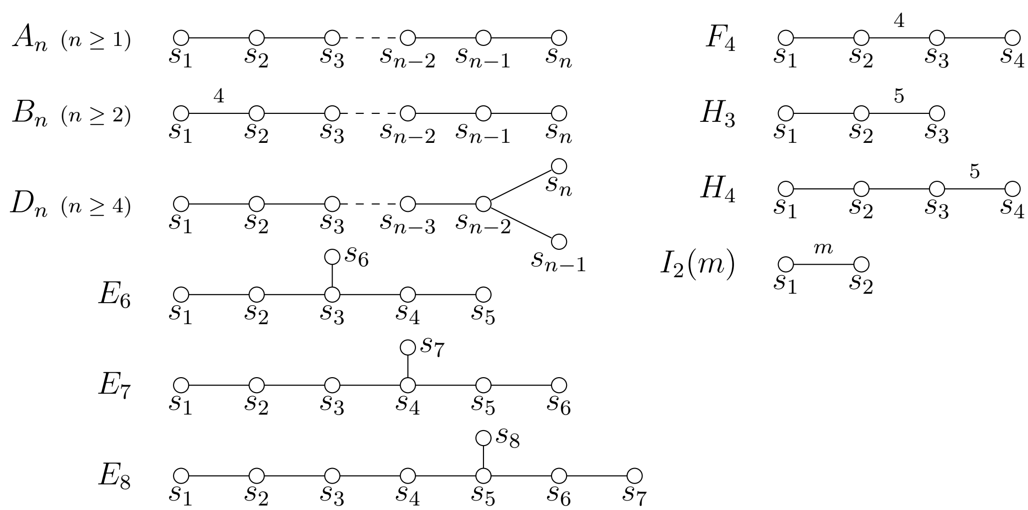


Figure 1.3: All irreducible finite Coxeter groups labelled with their type to the left. The index of the type refers to the number of nodes in the graph, *i.e.*, the number of simple reflections. Note that the type A_n Coxeter groups are also known as the symmetric groups S_{n+1}. Additionally, the type $I_2(m)$ Coxeter groups are the dihedral groups $\mathcal{D}_m$ of order $2m$.

In fact, thanks to the notion of Coxeter graphs, we have the following theorem on the classification of irreducible finite Coxeter groups.

Theorem 1.2.3 *(Coxeter, 1935, Theorem 1) The only irreducible finite Coxeter groups are those with Coxeter graphs in Figure 1.3.*

Therefore, by Theorem 1.2.1, every finite reflection group can be represented by a Coxeter graph as well. In the next section we detail how to represent a finite Coxeter group as a finite reflection group.

1.3 Geometric representation of Coxeter groups

In this section we describe the classical geometric representation of finite Coxeter groups as finite reflection groups due to Tits. Let (W, S) be a Coxeter system with W finite. We start with a real vector space V with basis $\Delta = \{\alpha_s \mid \ s \in S\}$. We impose a geometry on V in order to replicate "angles" between the vectors in Δ. To this end we define a symmetric bilinear form on the simple reflections such that for $s, t \in S$ we associate the angle between α_s and α_t with $m_{s,t}$, the order of s and t:

$$B(\alpha_s, \alpha_t) = -\cos\left(\frac{\pi}{m_{s,t}}\right).$$

In this case, since $m_{s,s} = 1$, we have $B(\alpha_s, \alpha_s) = -\cos(\pi) = 1$. In the case that $m_{s,t} = \infty$ we set $B(\alpha_s, \alpha_t) = -1$. For $s \in S$, the reflection $\sigma_{\alpha_s} : V \to V$ is the linear map:

$$\sigma_{\alpha_s}(v) = v - 2B(v, \alpha_s)\alpha_s, \text{ for all } v \in V.$$

With the vector space V, the form B and the linear maps σ_α we have the following proposition, see for instance (Humphreys, 1990, Proposition 5.3, Corollary 5.4, Theorem 6.4).

Theorem 1.3.1 *The linear representation $\sigma : W \to \mathrm{GL}(V)$ sending s to σ_{α_s} is faithful, $\sigma(W) \cong W$ and $\sigma(W)$ preserves the form B on V. Moreover, W is finite if and only if B is a positive-definite symmetric bilinear form*

The homomorphism σ is called the *geometric representation* of the Coxeter system (W, S). This representation is of great interest for understanding the elements of w written as words on S, as we explain now.

1.4 Length and reduced words

We return to the study of a Coxeter system (W, S). Since W is generated by the set of simple reflections S, the elements of W can be written as words over the alphabet S, *i.e.*, w can be written as a product of simple reflections in S. The *length* $\ell(w)$ of an element $w \in W$ is then the minimal length of the words for w as a product of generators in S:

$$\ell(w) = \min \{n \mid \; w = s_1 s_2 \ldots s_n, \; s_i \in S\}.$$

Given a word $w = s_1 s_2 \cdots s_k$ for $w \in W$ and $s_i \in S$, we say $s_1 s_2 \cdots s_k$ is a *reduced word of* w (or simply *reduced*[1]) if $k = \ell(w)$.

Although reduced words for a given element w must exist, they are not in general unique. As an example, let W be the type A_2 Coxeter group with simple reflections $S = \{s, t\}$ such that $(st)^3 = e$. One observes that the relation $(st)^3 = e$ can be rewritten by multiplying both sides by simple reflections (using the relations $s^2 = e$ for all $s \in S$) until the length of the word on either side of the equality

[1]In the literature, this is also referred to as a *reduced expression*.

is the same. Relations of the form $sts \ldots = tst \ldots$, with $m_{s,t}$ simple reflections on both sides, are known as *braid relations*. The braid relation $sts = tst$ implies that sts and tst are both two different reduced words of the same element in W.

Example 1.4.1 Let $S = \{s_1, s_2, s_3, s_4\}$ be the simple reflections for the A_4 Coxeter group. Recall that we have the relations:

$$(s_i)^2 = (s_1s_2)^3 = (s_1s_3)^2 = (s_1s_4)^2 = (s_2s_3)^3 = (s_2s_4)^2 = (s_3s_4)^3 = e$$

which can be deduced from the Coxeter graph of the type A_4 Coxeter group in Figure 1.3. To find a reduced word for $w \in W$ we can use our relations to reduce the number of elements in the word. For instance, let $w = s_1s_2s_4s_2s_3s_4s_3s_4s_1$. We aim to find a reduced word for w.

$$\begin{aligned}
w &= s_1s_2\underline{s_4s_2}s_3s_4s_3s_4s_1 && (s_4s_2 = s_2s_4)\\
&= s_1\underline{s_2s_2}s_4s_3s_4s_3s_4s_1 && (s_2s_2 = e)\\
&= s_1s_4\underline{s_3s_4s_3}s_4s_1 && (s_3s_4s_3 = s_4s_3s_4)\\
&= s_1\underline{s_4s_4}s_3\underline{s_4s_4}s_1 && (s_4s_4 = e)\\
&= s_1\underline{s_3s_1} && (s_3s_1 = s_1s_3)\\
&= \underline{s_1s_1}s_3 && (s_1s_1 = e)\\
&= s_3.
\end{aligned}$$

The relations on the right of the equations above are the relations used to go from one word to the next. In our case, since $w = s_1s_2s_4s_2s_3s_4s_3s_4s_1 = s_3$, then $\ell(w) = 1$ and $w = s_3$ is a reduced word for w.

Although we can use our relations to find a reduced expression of a given word, there is an easier way to find reduced expressions: using the geometric representation, which we present in the next section.

1.5 Root systems, reflections and inversion sets

In this section we survey root systems, which will be used to give an efficient way to compute the length of a word w in W.

Given a Coxeter system (W, S), we associate a root system Φ to (W, S) by taking $\Delta = \{\alpha_s \mid s \in S\}$ and generating Φ by acting on Δ by W:

$$\Phi = W(\Delta) = \{w(\alpha) \mid w \in W \text{ and } \alpha \in \Delta\}.$$

An example of a root system associated to the type A_2 Coxeter group is given in Figure 1.4. The elements in Φ are known as *roots* and we call the elements in Δ *simple roots*. The root system is split into *positive roots* Φ^+ and *negative roots* $\Phi^- = -\Phi^+$, by setting $\Phi^+ = \text{cone}(\Delta) \cap \Phi$ where cone is defined as in § 1.7.

It turns out that each positive root in Φ^+ is associated to a reflection in

$$T = \left\{wsw^{-1} \mid w \in W,\, s \in S\right\}.$$

See for instance (Björner & Brenti, 2005, Proposition 4.4.5).

Proposition 1.5.1 *The map $\rho : \Phi^+ \to T$ where $\rho(w(\alpha_s)) = wsw^{-1}$ for $w \in W$ and $s \in S$ is well-defined and is a bijection.*

We call T the *set of reflections*. By this proposition, for each $\alpha \in \Phi^+$ there is a reflection $t_\alpha \in T$ associated to α and for each $t = wsw^{-1} \in T$ there is an $\alpha_t \in \Phi^+$ associated to t where $\alpha_t = w(\alpha_s)$ if $w(\alpha_s) \in \Phi^+$ and $\alpha_t = -w(\alpha_s)$ otherwise.

Example 1.5.2 Let W be a Coxeter group of type A_2. We know that A_2 is the Coxeter group with two generators $S = \{s, t\}$ such that $(st)^3 = e$ which can be deduced from Figure 1.3. The group W turns out to have order 6 with elements $\{e, s, t, st, ts, sts\}$ and three reflections $T = \{s, t, sts\}$. The root system associated to W is given by $\Phi = \{\pm\alpha_s, \pm\alpha_t, \pm\alpha_{sts}\}$ where $\alpha_{sts} = s(\alpha_t) = t(\alpha_s) = \alpha_s + \alpha_t$.

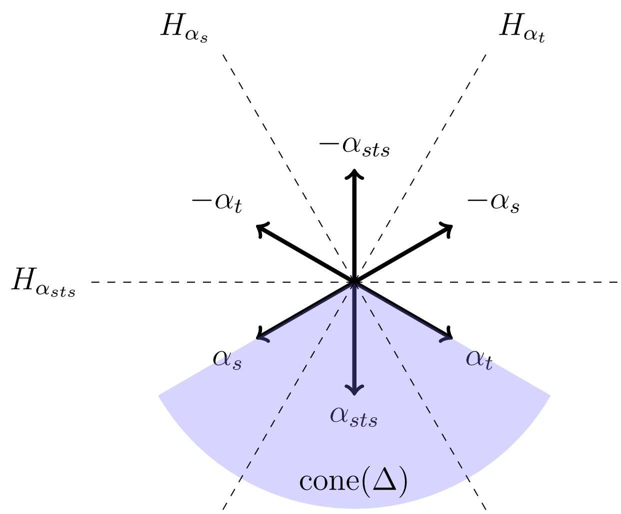


Figure 1.4: The root system associated to the type A_2 Coxeter group. The simple roots are given by $\Delta = \{\alpha_s, \alpha_t\}$ and the positive roots are given by $\Phi^+ = \{\alpha_s, \alpha_t, \alpha_{sts}\} = \mathrm{cone}(\Delta) \cap \Phi$.

We observe this in Figure 1.4 where we can also observe that the set Φ decomposes into a set of positive roots (contained in cone(Δ)) and a set of negative roots.

We now use the root system Φ to compute the length of $w \in W$. The *inversion set* of an element $w \in W$ is the set $N(w) := \Phi^+ \cap w\,(\Phi^-)$, which is the set of the positive roots that are sent to negatives roots by w^{-1}. If $w = uv = s_1 s_2 \cdots s_k$ is reduced with $u = s_1 \cdots s_i$ and $v = s_{i+1} \cdots s_k$ for some i, then $N(w)$ turns out to be:

$$N(w) = N(u) \sqcup u\,(N(v)) = \{\alpha_{s_1}, s_1\,(\alpha_{s_2})\,, \ldots, s_1 s_2 \cdots s_{k-1}\,(\alpha_{s_k})\}\,.$$

In particular, we have the following proposition, see for instance (Humphreys, 1990, Corollary 1.7).

Proposition 1.5.3 *For any element $w \in W$, the number of roots in the inversion set of w is the length of w, i.e., $\ell(w) = |N(w)|$.*

If W is finite, then it can be observed that there is always some element that sends every positive root to a negative one. We call this element the *longest element* and denote it by $w_\circ$. In fact, we have the following equalities

$$\ell(w_\circ) = |N(w_\circ)| = |\Phi^+| = |T|.$$

Example 1.5.4 We continue with the type A_2 Coxeter group whose root system can be found in Figure 1.4. We find the inversion set $N(st)$. To do this, we start with $\Phi^- = \{-\alpha_s, -\alpha_t, -\alpha_{sts}\}$ and reflect by st. First, we reflect over H_{α_t} to get $t(\Phi^-)$:

$$t(\Phi^-) = \{t(-\alpha_s),\, t(-\alpha_t),\, t(-\alpha_{sts})\} = \{-\alpha_{sts},\, \alpha_t,\, -\alpha_s\}\,.$$

Next, we reflect over H_{α_s} to get $st(\Phi^-)$:

$$st(\Phi^-) = s\left(t(\Phi^-)\right) = \{s(-\alpha_{sts}),\, s(\alpha_t),\, s(-\alpha_s)\} = \{-\alpha_t,\, \alpha_{sts},\, \alpha_s\}\,.$$

Then

$$N(st) = \Phi^+ \cap st(\Phi^-) = \Phi^+ \cap \{-\alpha_t,\, \alpha_{sts},\, \alpha_s\} = \{\alpha_{sts},\, \alpha_s\}\,.$$

As we can observe, $\ell(st) = |N(st)| = 2$.

Computing the inversion set for every element in the same way, we get:

$$\begin{array}{ccc} N(e) = \varnothing & N(t) = \{\alpha_t\} & N(ts) = \{\alpha_t,\, \alpha_{sts}\} \\ N(s) = \{\alpha_s\} & N(st) = \{\alpha_s,\, \alpha_{sts}\} & N(sts) = \{\alpha_s,\, \alpha_t,\, \alpha_{sts}\} = \Phi^+. \end{array}$$

Notice that $N(w_\circ) = N(sts) = \Phi^+$. The inversion sets ordered by inclusion are found in Figure 1.5.

We next survey a combinatorial description of the poset of inversion sets ordered by inclusion.

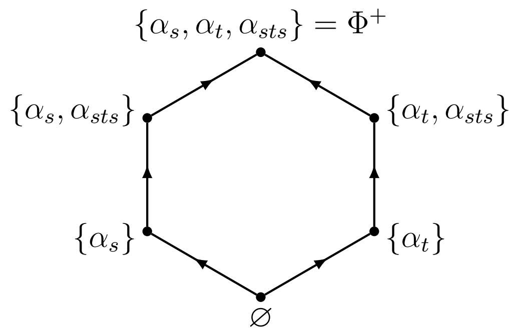


Figure 1.5: The inversion sets of the type A_2 Coxeter group ordered by inclusion.

1.6 Weak order

In this section we survey the (right) weak order on a Coxeter system (W, S) which is a combinatorial interpretation of the poset of inversion sets ordered by inclusion. For a background on orders, posets or lattices, the reader is invited to read the brief introduction provided in Appendix A.

An element u is a *prefix* of v if there exists some reduced word for u that is a prefix of some reduced word for v, *i.e.*, $\ell(u^{-1}v) = \ell(v) - \ell(u)$. A *suffix* of v is defined similarly, *i.e.*, $\ell(vu^{-1}) = \ell(v) - \ell(u)$. The *right weak order* is the order $\leq_R$ on W such that for $u, v \in W$,

$$u \leq_R v \iff u \text{ is a prefix of } v.$$

The left weak order is defined similarly with suffixes instead of prefixes. Since we exclusively use the right weak order we will say *weak order* when referring to the right weak order.

Example 1.6.1 An example of the weak order can be found in Figure 1.6 for the types A_2 and B_2 Coxeter groups. Notice that in both these Coxeter groups the element s and the element ts are incomparable. This is because there is no reduced word of ts which begins with s.

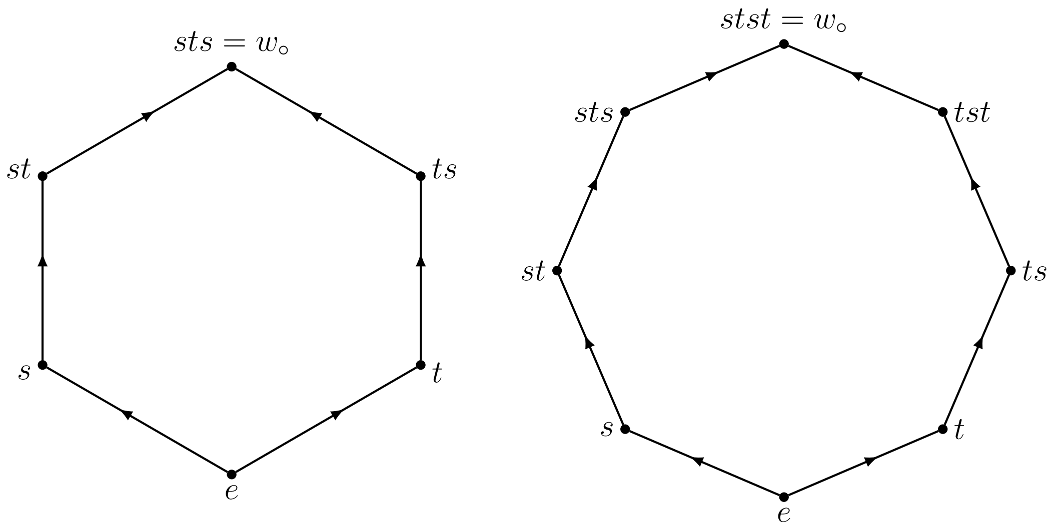


Figure 1.6: The weak order for the types A_2 and B_2 Coxeter groups.

Example 1.6.2 As another example, let $u = s_1$ and $v = s_2s_1s_2$ in the type A_3 Coxeter group with $S = \{s_1, s_2, s_3\}$. We might assume that u is not a prefix of v since $s_2s_1s_2$ does not begin with $s_1 = u$ and $s_2s_1s_2$ is a reduced word for v. But our definition says that u can be a prefix of *any* reduced word of v. Recalling that we have the braid relation $s_2s_1s_2 = s_1s_2s_1$ then $s_1s_2s_1$ is another reduced word of v and is one which has $u = s_1$ as a prefix. Therefore $u \leq_R v$ in the weak order.

In 1984 Björner stated that the weak order $(W, \leq_R)$ is a lattice for any finite Coxeter group and a meet-semilattice for any Coxeter group.

Theorem 1.6.3 *(Björner, 1984, Theorem 8) The weak order of a Coxeter group W is a meet-semilattice. Furthermore, if W is finite, then the weak order is a lattice.*

A proof of this fact can be found in (Björner et al., 1990).

The poset of inversion sets ordered by inclusion turns out to be isomorphic with the weak order poset. This is a classical result from (Björner, 1984, Proposition 2), see for instance the discussion and references within (Hohlweg & Labbé, 2016).

Proposition 1.6.4 *Let $\mathscr{B} = \{N(w) \mid \ w \in W\}$ be the set of all inversion sets for a finite Coxeter group W. Then the map $(W, \leq_R) \to (\mathscr{B}, \subseteq)$ is a poset isomorphism. In other words, for $u, v \in W$, $u \leq_R v$ if and only if $N(u) \subseteq N(v)$.*

As an example, one can compare the inversion sets ordered by inclusion in Figure 1.5 with the weak order on the left in Figure 1.6. By this result, for W finite, the longest element $w_\circ$ is the top element in the weak order and is the element of maximal length in W. The weak order has a geometric interpretation as the 1-skeleton of a polytope named the permutahedron, which we study in the next section.

1.7 Permutahedron

Let (W, S) be a Coxeter system. In this section we study the W-permutahedron, a polytope that is generated by the action of W on a point.

Recall that, given a real vector space V, a *polytope* P is the convex hull of finitely many points of V or, equivalently, the bounded intersection of finitely many affine halfspaces of V. The *faces* $\mathscr{F}_P$ of a polytope P (or simply $\mathscr{F}$ if there is no ambiguity) is the set of intersections of P with its supporting hyperplanes. The faces of dimension 0 are called *vertices* and the faces of codimension 1 are called *facets*. The *face lattice* $(\mathscr{F}_P, \subseteq)$ of P is the lattice where the faces $\mathscr{F}_P$ are ordered by inclusion giving us a way to encode the relationship between the faces of a polytope. As the name implies, this partial order is a lattice with the polytope itself as the top element, the empty face as the bottom element, the vertices of the polytope as the atoms and the facets as the coatoms.

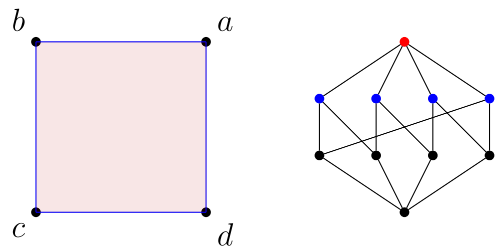


Figure 1.7: The square on the left is the polytope which is acquired by taking the convex hull of the four points $\{a, b, c, d\}$. On the right is the face lattice associated to the square.

Example 1.7.1 An example of a polytope is given in Figure 1.7. Let P be the polytope on the left in Figure 1.7 which is the convex hull of the four points $\{a, b, c, d\}$ in Figure 1.7. On the right hand side of Figure 1.7 is the face lattice associated to the faces of P. The bottom element in the face lattice is the empty set, the first row contains the vertices, the second row contains the edges and the row on top in the face lattice represents the square itself.

Given a finite Coxeter system (W, S) action on the vector space V as described in § 1.3, the *W-permutahedron* is the convex hull of the orbit under W of some generic point $v \in V$. Here *generic* means that v is not contained on any reflection hyperplane of W. The W-permutahedron is given by:

$$\mathsf{Perm}^v(W) = \operatorname{conv}\{w(v) \mid w \in W\}.$$

Example 1.7.2 An example of the W-permutahedron in type A_2 is given in Figure 1.8 where the point v has been reflected over the three hyperplanes. The hexagon is then the convex hull of the six points produced under this W-action. The face lattice of the W-permutahedron is given in Figure 1.9, where the atoms are the 6 vertices and the coatoms are the 6 edges.

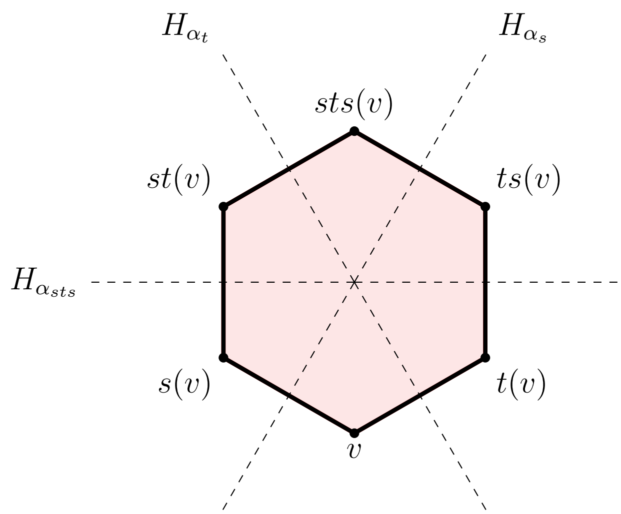


Figure 1.8: The W-permutahedron $\mathsf{Perm}^v(W)$ of a generic point $v \in V$ for the Coxeter group of type A_2.

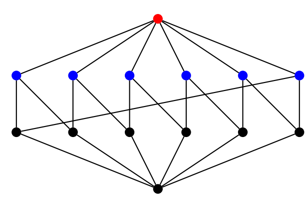


Figure 1.9: The face lattice $(\mathscr{F}, \subseteq)$ of the W-permutahedron of the type A_2 Coxeter group W.

Recall that a *cone* $\text{cone}(Y)$ generated by a nonempty set of vectors $Y \subseteq V$ is the set of all finite nonnegative linear combinations of vectors of Y, *i.e.*,

$$\text{cone}(Y) = \{\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_k v_k \mid \lambda_i \in \mathbb{R}_{\geq 0},\, v_i \in Y,\, k \geq 1\}$$

For a face $F \in \mathscr{F}$ of a polytope P, the *inner primal cone* of F is the cone generated by $\{u - v \mid u \in P,\, v \in F\}$. The inner primal cone is the cone at the face F with vectors pointing "inside" the polytope P. The *outer normal cone* of F is the cone generated by the outer normal vectors of the facets of P which contain F. These

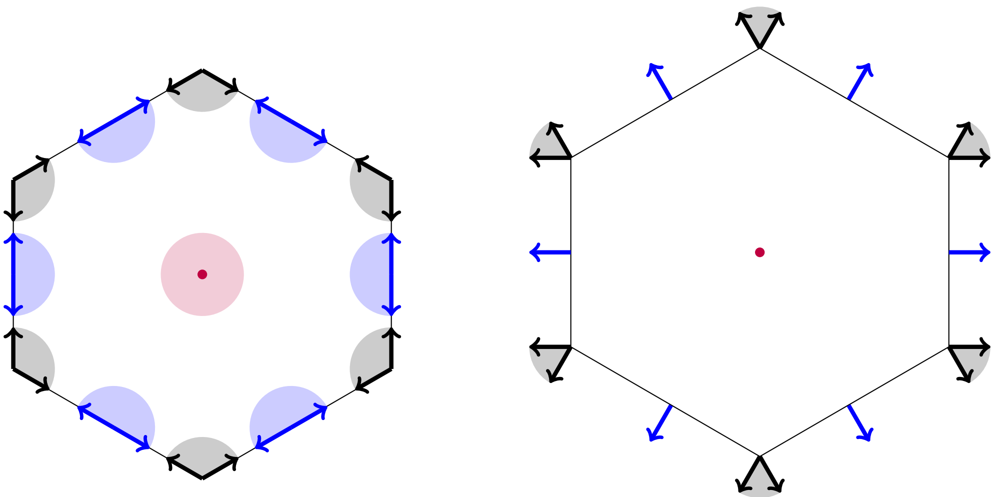


Figure 1.10: For the W-permutahedron of type A_2, the figure on the left is the inner primal cones of each face and on the right is the outer normal cones of each face.

two cones are particularly interesting as they are polar to one another. For a cone C, the *polar cone* is given by:

$$C^{\diamond} = \{u \in V^{\star} \mid \langle u, v \rangle \leq 0 \text{ for all } v \in C\} .$$

Example 1.7.3 Consider the type A_2 Coxeter group W and let $\mathsf{Perm}^v(W)$ be its associated W-permutahedron as in Figure 1.8. The inner primal cones and the outer normal cones of $\mathsf{Perm}^v(W)$ are given in Figure 1.10. The inner primal cones (left) are the cones at each face which go towards the inside of the polytope. On the other hand, the outer normal cones (right) are the cones which are generated by the normal vectors to the hyperplanes supporting each face.

1.8 Parabolic subgroups and cosets

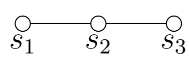
Given a finite Coxeter system (W, S), the faces of the W-permutahedron encode an important class of subgroups of W and their cosets: the set of standard parabolic cosets associated to standard parabolic subgroups of W.

Notice that any subset I of S generates another Coxeter group W_I with simple roots $\Delta_I := \{\alpha_s \mid s \in I\}$, root system $\Phi_I = W_I(\Delta_I)$ and longest element $w_{\circ,I}$. We call $W_I = \langle I \rangle$ a *standard parabolic subgroup* of W. See for instance (Humphreys, 1990, Theorems 1.10 and 5.5).

Theorem 1.8.1 *For (W, S) a Coxeter system and $I \subseteq S$. Let $W_I = \langle I \rangle$ and $\Delta_I = \{\alpha_s \mid s \in I\} \subseteq \Delta$. Then (W_I, I) is a Coxeter system and Φ_I, the intersection of Φ with the $\mathbb{R}$-span of Δ_I, is a root system. Furthermore, the length function ℓ_I of W_I is the same length function as in W, i.e., $\ell_I = \ell$.*

Another way to describe W_I is to start with the Coxeter graph of W and restrict it to the subgraph with vertex set I; giving the Coxeter graph to the group W_I. As extreme examples $W_\varnothing = \{e\}$ and $W_S = W$.

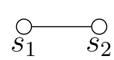
Example 1.8.2 Let $S = \{s_1, s_2, s_3\}$ be the set of simple reflections for the Coxeter group W of type A_3 whose Coxeter graph is given by:



Then $S' = \{s_1, s_2\}$ generates

$$W_{S'} = \left\langle S' \mid (s_1)^2 = (s_2)^2 = (s_1 s_2)^3 = e \right\rangle,$$

which is a Coxeter group of type A_2 whose Coxeter graph is given by:



Letting $S'' = \{s_1, s_3\}$, then S'' generates

$$W_{S''} = \left\langle S'' \mid (s_1)^2 = (s_3)^2 = (s_1 s_3)^2 = e \right\rangle$$

which is a Coxeter group of type $A_1 \times A_1$ whose Coxeter graph is given by:

To describe the cosets of a standard parabolic subgroup, we consider the set

$$W^I := \{w \in W \mid \ \ell(ws) > \ell(w) \text{ for all } s \in I\}.$$

For the subgroups $W_\varnothing = \{e\}$ and $W_S = W$ we have $W^\varnothing = W$ and $W^S = \{e\}$ respectively. In fact, given any standard parabolic subgroup W_I and any element $w \in W$, then w has a unique factorization relative to W_I, see for instance (Björner & Brenti, 2005, Proposition 2.4.4).

Proposition 1.8.3 *Let $I \subseteq S$, then for every $w \in W$, w has a unique factorization $w = w^I \cdot w_I$ where $w^I \in W^I$ and $w_I \in W_I$. Furthermore, $\ell(w) = \ell(w^I) + \ell(w_I)$. In particular, W/W_I and W^I are in bijection.*

Therefore, W^I is the set of *minimal length coset representatives* of the coset W/W_I. A *standard parabolic coset* is a coset of the form xW_I with $x \in W^I$. Any standard parabolic coset xW_I forms an interval $[x, xw_{\circ,I}]_R$ in the weak order, since $\ell(w) = \ell(w^I) + \ell(w_I)$. We call these intervals *facial intervals* associated to the standard parabolic cosets in W/W_I.

Example 1.8.4 For the A_2 Coxeter group W we have $S = \{s, t\}$. There are four standard parabolic subgroups for S: $W_\varnothing$, $W_{\{s\}}$, $W_{\{t\}}$ and W_S.

For W_S, since $W_S = W$ the only coset representative is e. Notice that since W_S contains every element, it is represented by the facial interval $[e, w_\circ]_R$ which is the entire weak order.

For $W_{\{s\}}$, there are only three (minimal length)coset representatives that we can have, $W^{\{s\}} = \{e, t, st\}$. Since $W_{\{s\}}$ contains two elements (e and s), the facial intervals we get for each coset are:

$$eW_{\{s\}} \leftrightarrow [e,s]_R \quad tW_{\{s\}} \leftrightarrow [t,ts]_R \quad stW_{\{s\}} \leftrightarrow [st,sts]_R.$$

It can be observed that we get similar results with $W_{\{t\}}$.

The subgroup $W_\varnothing$ contains the unique element e, *i.e.*, $W_\varnothing = \{e\}$. Therefore every element of the group W will be a minimal coset representative and the facial intervals associated to each coset will be the singletons: $[e,e]_R$, $[s,s]_R$, $[t,t]_R$, $[st,st]_R$, $[ts,ts]_R$ and $[sts,sts]_R$.

Let $\mathcal{P}_W$ denote the set of all standard parabolic cosets:

$$\mathcal{P}_W = \left\{ xW_I \mid \ I \subseteq S,\, x \in W^I \right\}.$$

The set $\mathcal{P}_W$ is called the *Coxeter complex* of W. To each standard parabolic coset in $\mathcal{P}_W$ we can associate a face of the W-permutahedron through the bijective map $\mathbf{F} : \mathcal{P}_W \to \mathsf{Perm}^v(W)$ where $\mathbf{F}(xW_I) = x\left(\mathsf{Perm}^v(W_I)\right) = \mathsf{Perm}^{x(v)}(xW_Ix^{-1})$. Therefore each k-dimensional face of $\mathsf{Perm}^v(W)$ is associated to a standard parabolic coset xW_I with $|I| = k$. Examples in types A_2 and B_2 are given in Figure 1.11.

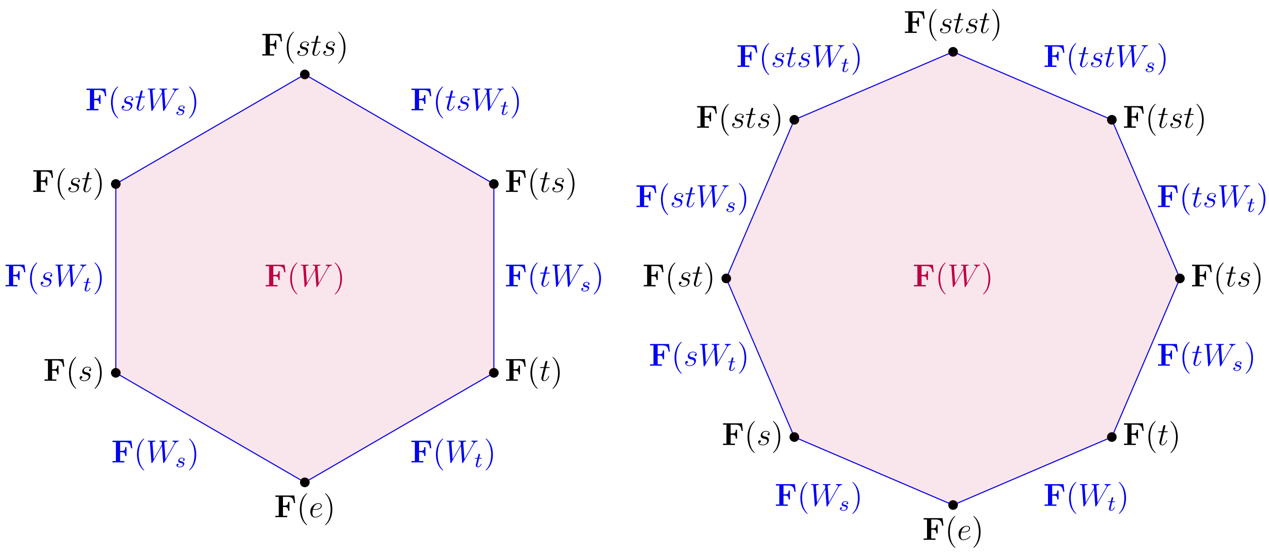


Figure 1.11: Standard parabolic cosets of the type A_2 and B_2 Coxeter groups and the corresponding faces on their permutahedra.

CHAPTER II

THE FACIAL WEAK ORDER AND ITS LATTICE QUOTIENTS

The text in this chapter was published in *Transactions of the American Mathematical Society* in 2018 and was written by myself, Christophe Hohlweg and Vincent Pilaud.

The (right) Cayley graph of a Coxeter system (W, S) is naturally oriented by the (right) weak order on W: an edge is oriented from w to ws if $s \in S$ is such that $\ell(w) < \ell(ws)$, see (Björner & Brenti, 2005, Chapter 3) for details. A celebrated result of A. Björner (Björner, 1984) states that the weak order is a complete meet-semilattice and even a complete ortholattice in the case of a finite Coxeter system. The weak order is a very useful tool to study Coxeter groups as it encodes the combinatorics of reduced words associated to (W, S), and it underlines the connection between the words and the root system via the notion of inversion sets, see for instance (Dyer, 2011; Hohlweg & Labbé, 2016) and the references therein.

In the case of a finite Coxeter system, the Cayley graph of W is isomorphic to the 1-skeleton of the W-permutahedron. Then the weak order is given by an orientation of the 1-skeleton of the W-permutahedron associated to the choice of a linear form of the ambient Euclidean space. This point of view was very useful in order to build generalized associahedra out of a W-permutahedron using N. Reading's Cambrian lattices, see (Reading, 2012; Hohlweg et al., 2011; Hohlweg, 2012a).

In this paper, we study a poset structure on all faces of the W-permutahedron that we call the (right) *facial weak order*. This order was introduced by D. Krob, M. Latapy, J.-C. Novelli, H.-D. Phan and S. Schwer in (Krob et al., 2001) for the symmetric group then extended by P. Palacios and M. Ronco in (Palacios & Ronco, 2006) for arbitrary finite Coxeter groups. Recall that the faces of the W-permutahedron are naturally parameterized by the Coxeter complex $\mathcal{P}_W$ which consists of all standard parabolic cosets W/W_I for $I \subseteq S$. The aims of this article are:

(1) To give two alternative characterizations of the facial weak order (see Theorem (2.2.14)): one in terms of root inversion sets of parabolic cosets which extend the notion of inversion sets of elements of W, and the other one as the subposet of the poset of intervals of the weak order induced by the standard parabolic cosets. The advantage of these two definitions is that they give immediate global comparison, while the original definition of (Palacios & Ronco, 2006) uses cover relations.

(2) To show that the facial weak order is a lattice (see Theorem (2.2.19)), whose restriction to the vertices of the permutahedron produces the weak order as a sublattice. This result was motivated by the special case of type A proved in (Krob et al., 2001).

(3) To discuss generalizations of these statements to infinite Coxeter groups via the Davis complex (see Theorem (2.2.30)).

(4) To show that any lattice congruence $\equiv$ of the weak order naturally extends to a lattice congruence $\boxed{\equiv}$ of the facial weak order (see Theorem (2.3.11)). This provides a complete description (see Theorem (2.3.22)) of the fan $\mathcal{F}_\equiv$ associated to the weak order congruence $\equiv$ in N. Reading's work (Reading, 2005): while the classes of $\equiv$ correspond to maximal cones in $\mathcal{F}_\equiv$, the classes

> of $\equiv$ correspond to all cones in $\mathcal{F}_{\equiv}$. Relevant illustrations are given for Cambrian lattices and fans (Reading, 2006; Reading & Speyer, 2009), which extend to facial Cambrian lattices on the faces of generalized associahedra (see Theorem (2.3.30)).

The results of this paper are based on combinatorial properties of Coxeter groups, parabolic cosets, and reduced words. However, their motivation and intuition come from the geometry of the Coxeter arrangement and of the W-permutahedron. We made a point to introduce enough of the geometrical material to make the geometric intuition clear.

2.1 Preliminaries

We start by fixing notations and classical definitions on finite Coxeter groups. Details can be found in textbooks by J. Humphreys (Humphreys, 1990) and A. Björner and F. Brenti (Björner & Brenti, 2005). The reader familiar with finite Coxeter groups and root systems is invited to proceed directly to Section (2.2).

2.1.1 Finite reflection groups and Coxeter systems

Let $(V, \langle \cdot \mid \cdot \rangle)$ be an n-dimensional Euclidean vector space. For any vector $v \in V \smallsetminus \{0\}$, we denote by s_v the reflection interchanging v and $-v$ while fixing the orthogonal hyperplane pointwise. Remember that $ws_v = s_{w(v)}w$ for any vector $v \in V \smallsetminus \{0\}$ and any orthogonal transformation w of V.

We consider a *finite reflection group* W acting on V, that is, a finite group generated by reflections in the orthogonal group $O(V)$. The *Coxeter arrangement* of W is the collection of all reflecting hyperplanes. Its complement in V is a union of open polyhedral cones. Their closures are called *chambers*. The *Coxeter fan* is the polyhedral fan formed by the chambers together with all their faces. This fan is *complete* (its cones cover V) and *simplicial* (all cones are simplicial), and we can assume without loss of generality that it is *essential* (the intersection of all chambers is reduced to the origin). We fix an arbitrary chamber $\mathcal{C}$ which we call the *fundamental chamber*. The n reflections orthogonal to the facet defining hyperplanes of $\mathcal{C}$ are called *simple reflections*. The set S of simple reflections generates W. The set of *reflections* is the set $T = \{wsw^{-1} \mid w \in W \text{ and } s \in S\}$. The pair (W, S) forms a *Coxeter system*. See Figure (2.1) for an illustration of the Coxeter arrangements of types A_3, B_3, and H_3.

2.1.2 Roots and weights

We consider a *root system* Φ for W, *i.e.*, a set of vectors invariant under the action of W and containing precisely two opposite roots orthogonal to each reflecting hyperplane of W. The *simple roots* Δ are the roots orthogonal to the defining hyperplanes of $\mathcal{C}$ and pointing towards $\mathcal{C}$. They form a linear basis of V. The root system Φ splits into the *positive roots* $\Phi^+ := \Phi \cap \operatorname{cone}(\Delta)$ and the *negative*

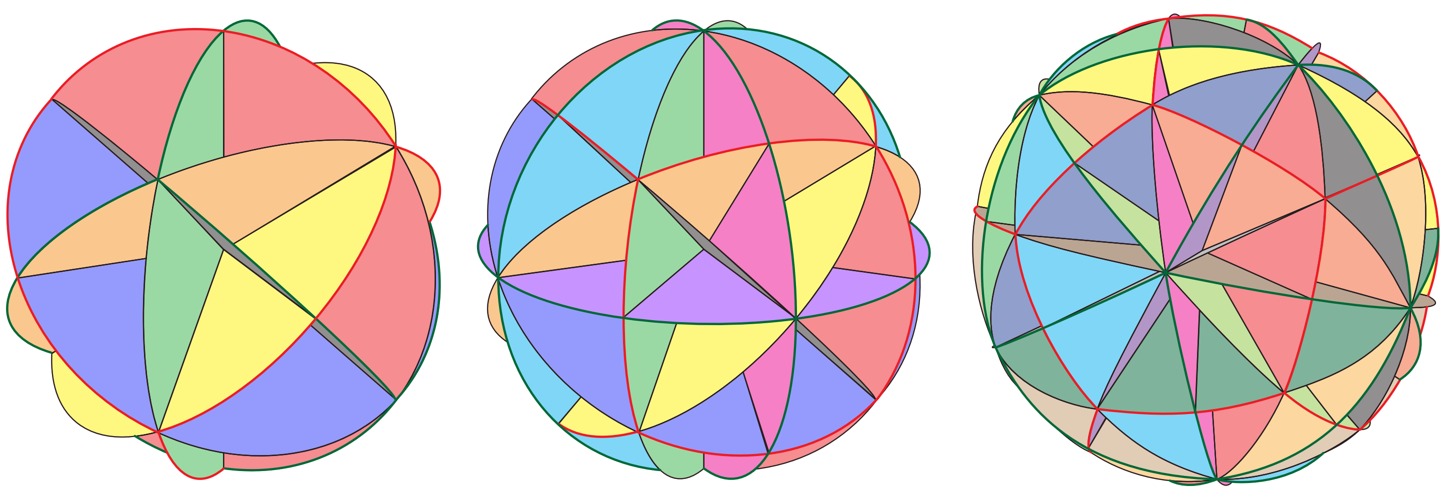


Figure 2.1: The type A_3, B_3, and H_3 Coxeter arrangements.

roots $\Phi^- := \Phi \cap \mathrm{cone}(-\Delta) = -\Phi^+$, where $\mathrm{cone}(X)$ denotes the set of nonnegative linear combinations of vectors in $X \subseteq V$. In other words, the positive roots are the roots whose scalar product with any vector of the interior of the fundamental chamber $\mathcal{C}$ is positive, and the simple roots form the basis of the cone generated by Φ^+. Each reflecting hyperplane is orthogonal to one positive and one negative root. For a reflection $s \in T$, we set α_s to be the unique positive root orthogonal to the reflecting hyperplane of s, *i.e.*, such that $s = s_{\alpha_s}$.

We denote by $\alpha_s^\vee := 2\alpha_s / \langle\, \alpha_s \mid \alpha_s \,\rangle$ the *coroot* corresponding to $\alpha_s \in \Delta$, and by $\Delta^\vee := \{\alpha_s^\vee \mid\ s \in S\}$ the coroot basis. The vectors of its dual basis $\nabla := \{\omega_s \mid\ s \in S\}$ are called *fundamental weights*. In other words, the fundamental weights of W are defined by $\langle\, \alpha_s^\vee \mid \omega_t \,\rangle = \delta_{s=t}$ for all $s, t \in S$. Geometrically, the fundamental weight ω_s gives the direction of the ray of the fundamental chamber $\mathcal{C}$ not contained in the reflecting hyperplane of s. We let $\Omega := W(\nabla) = \{w(\omega_s) \mid\ w \in W, s \in S\}$ denote the set of all weights of W, obtained as the orbit of the fundamental weights under W.

2.1.3 Length, reduced words and weak order

The *length* $\ell(w)$ of an element $w \in W$ is the length of the smallest word for w as a product of generators in S. A word $w = s_1 \cdots s_k$ with $s_1, \dots, s_k \in S$ is called *reduced* if $k = \ell(w)$. For $u, v \in W$, the product uv is said to be *reduced* if the concatenation of a reduced word for u and of a reduced word for v is a reduced word for uv, *i.e.*, if $\ell(uv) = \ell(u) + \ell(v)$. We say that $u \in W$ is a *prefix* of $v \in W$ if there is a reduced word for u that is the prefix of a reduced word for v, *i.e.*, if $\ell(u^{-1}v) = \ell(v) - \ell(u)$.

The (right) *weak order* is the order on W defined equivalently by

$$u \le v \iff \ell(u) + \ell(u^{-1}v) = \ell(v) \iff u \text{ is a prefix of } v.$$

A. Björner shows in (Björner, 1984) that the weak order defines a lattice structure on W (finite Coxeter group), with minimal element e and maximal element $w_\circ$ (which sends all positive roots to negative ones and all positive simple roots to negative simple ones). The conjugation $w \mapsto w_\circ w w_\circ$ defines a weak order automorphism while the left and right multiplications $w \mapsto w_\circ w$ and $w \mapsto w w_\circ$ define weak order anti-automorphisms. We refer the reader to (Björner & Brenti, 2005, Chapter 3) for more details.

The weak order encodes the combinatorics of reduced words and enjoys a useful geometric characterization within the root system, which we explain now. The (left) *inversion set* of w is the set $\mathbf{N}(w) := \Phi^+ \cap w(\Phi^-)$ of positive roots sent to negative ones by w^{-1}. If $w = uv$ is reduced then $\mathbf{N}(w) = \mathbf{N}(u) \sqcup u\Big(\mathbf{N}(v)\Big)$. In particular, we have $\mathbf{N}(w) = \Big\{\alpha_{s_1}, s_1(\alpha_{s_2}), \dots, s_1 s_2 \cdots s_{k-1}(\alpha_{s_k})\Big\}$ for any reduced word $w = s_1 \cdots s_k$, and therefore $\ell(w) = |\,\mathbf{N}(w)|$. Moreover, the weak order is characterized in term of inversion sets by:

$$u \le v \iff \mathbf{N}(u) \subseteq \mathbf{N}(v),$$

for any $u, v \in W$. We refer the reader to (Hohlweg & Labbé, 2016, Section 2) and the references therein for more details on inversion sets and the weak order.

We say that $s \in S$ is a *left ascent* of $w \in W$ if $\ell(sw) = \ell(w) + 1$ and a *left descent* of w if $\ell(sw) = \ell(w) - 1$. We denote by $D_L(w)$ the set of left descents of w. Note that for $s \in S$ and $w \in W$, we have $s \in D_L(w) \iff \alpha_s \in \mathbf{N}(w) \iff s \le w$. Similarly, $s \in S$ is a *right descent* of $w \in W$ if $\ell(ws) = \ell(w) - 1$, and we denote by $D_R(w)$ the set of right descents of w.

2.1.4 Parabolic subgroups and cosets

Consider a subset $I \subseteq S$. The *standard parabolic subgroup* W_I is the subgroup of W generated by I. It is also a Coxeter group with simple generators I, simple roots $\Delta_I \coloneqq \{\alpha_s \mid s \in I\}$, root system $\Phi_I = W_I(\Delta_I) = \Phi \cap \operatorname{span}(\Delta_I)$, length function $\ell_I = \ell|_{W_I}$, longest element $w_{\circ,I}$, etc. For example, $W_\varnothing = \{e\}$ while $W_S = W$.

We denote by $W^I \coloneqq \{w \in W \mid \ell(ws) > \ell(w) \text{ for all } s \in I\}$ the set of elements of W with no right descents in I. For example, $W^\varnothing = W$ while $W^S = \{e\}$. Observe that for any $x \in W^I$, we have $x(\Delta_I) \subseteq \Phi^+$ and thus $x(\Phi_I^+) \subseteq \Phi^+$. We will use this property repeatedly in this paper.

Any element $w \in W$ admits a unique factorization $w = w^I \cdot w_I$ with $w^I \in W^I$ and $w_I \in W_I$, and moreover, $\ell(w) = \ell(w^I) + \ell(w_I)$ (see (Björner & Brenti, 2005, Proposition 2.4.4)). Therefore, W^I is the set of *minimal length coset representatives* of the cosets W/W_I. Throughout the paper, we will always implicitly assume that $x \in W^I$ when writing that xW_I is a *standard parabolic coset*. Note that any standard parabolic coset $xW_I = [x, xw_{\circ,I}]$ is an interval in the weak order. The *Coxeter complex* $\mathcal{P}_W$ is the simplicial complex whose faces are all standard parabolic cosets of W

$$\mathcal{P}_W = \bigcup_{I \subseteq S} W/W_I = \Big\{xW_I \mid I \subseteq S,\ x \in W\Big\} = \Big\{xW_I \mid I \subseteq S,\ x \in W^I\Big\}$$

We will also need *Deodhar's Lemma*: for $s \in S$, $I \subseteq S$ and $x \in W^I$, either $sx \in W^I$ or $sx = xr$ for some $r \in I$. See for instance (Geck & Pfeiffer, 2000, Lemma 2.1.2) where it is stated for the cosets $W_I \backslash W$ instead of W/W_I.

2.1.5 Permutahedron

Remember that a *polytope* P is the convex hull of finitely many points of V, or equivalently a bounded intersection of finitely many affine halfspaces of V. The *faces* of P are the intersections of P with its supporting hyperplanes and the *face lattice* of P is the lattice of its faces ordered by inclusion. The *inner primal cone* of a face F of P is the cone generated by $\{u - v \mid u \in P, v \in F\}$. The *outer normal cone* of a face F of P is the cone generated by the outer normal vectors of the facets of P containing F. Note that these two cones are polar to each other. The *normal fan* is the complete polyhedral fan formed by the outer normal cones of all faces of P. We refer to (Ziegler, 1995) for details on polytopes, cones, and fans.

The *W-permutahedron* $\mathsf{Perm}^p(W)$ is the convex hull of the orbit under W of a generic point $p \in V$ (not located on any reflection hyperplane of W). Its vertex and facet descriptions are given by

$$\mathsf{Perm}^p(W) = \operatorname{conv}\Big\{w(p) \mid w \in W\Big\} = \bigcap_{\substack{s \in S \\ w \in W}} \Big\{v \in V \mid \langle w(\omega_s) \mid v \rangle \leq \langle \omega_s \mid p \rangle\Big\}.$$

Examples in types A_2 and B_2 are represented in Figure (2.2). Examples in types A_3, B_3, and H_3 are represented in Figure (2.3).

We often write $\mathsf{Perm}(W)$ instead of $\mathsf{Perm}^p(W)$ as the combinatorics of the W-permutahedron is independent of the choice of the point p and is encoded by the Coxeter complex $\mathcal{P}_W$. More precisely, each standard parabolic coset xW_I corresponds to a face $\mathbf{F}(xW_I)$ of $\mathsf{Perm}^p(W)$ given by

$$\mathbf{F}(xW_I) = x\Big(\mathsf{Perm}^p(W_I)\Big) = \mathsf{Perm}^{x(p)}\Big(xW_Ix^{-1}\Big).$$

Therefore, the k-dimensional faces of $\mathsf{Perm}^p(W)$ correspond to the cosets xW_I with $|I| = k$ and the face lattice of $\mathsf{Perm}^p(W)$ is isomorphic to the inclusion poset $(\mathcal{P}_W, \subseteq)$. The normal fan of $\mathsf{Perm}^p(W)$ is the Coxeter fan. The graph of

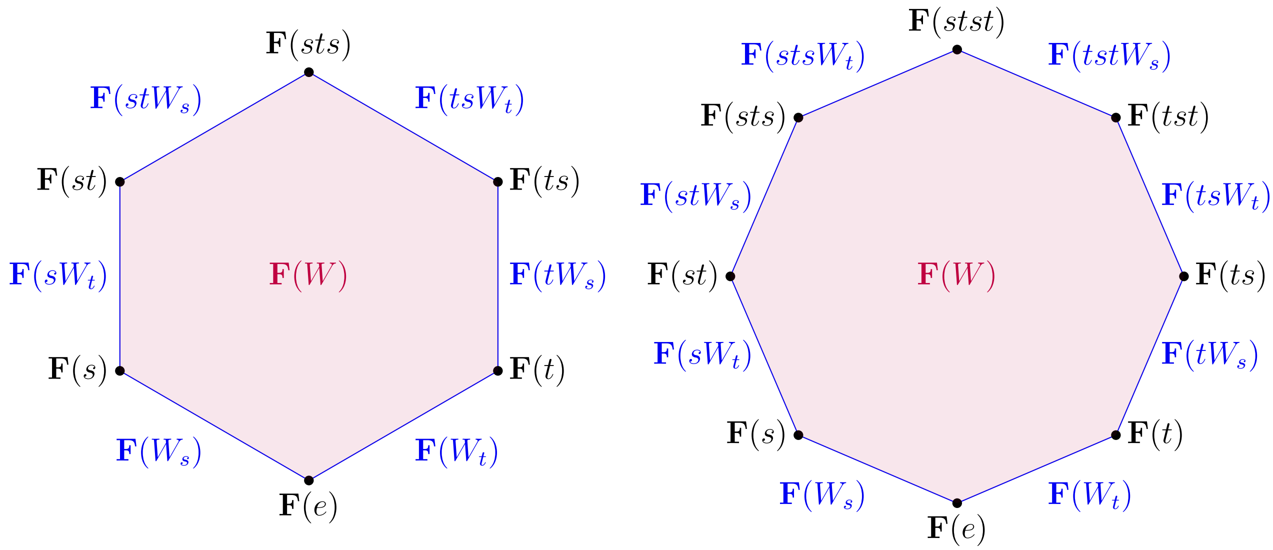


Figure 2.2: Standard parabolic cosets of the type A_2 and B_2 Coxeter groups and the corresponding faces on their permutahedra.

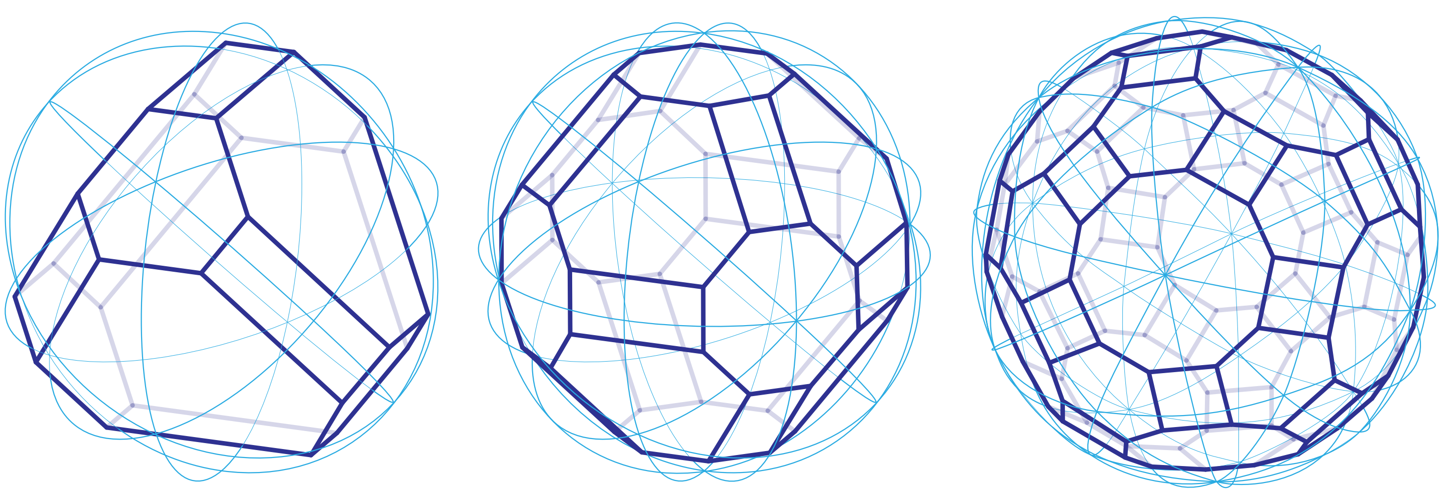


Figure 2.3: The type A_3, B_3, and H_3 permutahedra.

the permutahedron $\mathsf{Perm}^p(W)$ is isomorphic to the Cayley graph of the Coxeter system (W, S). Moreover, when oriented in the linear direction $w_\circ(p) - p$, it coincides with the Hasse diagram of the (right) weak order on W. We refer the reader to (Hohlweg, 2012a) for more details on the W-permutahedron.

Example 2.1.1 The Coxeter group of type A_{n-1} is the symmetric group $\mathfrak{S}_n$. Its simple generators are the simple transpositions $\tau_i = (i\ i+1)$ for $i \in [n-1]$ with relations $\tau_i^2 = 1$ and $\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}$. Its elements are permutations of $[n]$ and its standard parabolic cosets are ordered partitions of $[n]$. A root system for $\mathfrak{S}_n$ consists in the set of vectors $\{e_i - e_j \mid i \neq j \in [n]\}$ where $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbb{R}^n$. Note that, this construction does not give us an essential Coxeter fan. The type A_3 Coxeter arrangement is represented in Figure (2.1) (left), and the type A_2 and A_3 permutahedra are represented in Figures (2.2) (left) and (2.3) (left).

2.2 Facial weak order

In this section we study an analogue of the weak order on standard parabolic cosets. Note that the set $\mathcal{I}(P) := \{[x, X] \mid x, X \in P,\ x \leq X\}$ of all intervals of a poset P is itself ordered by $[x, X] \leq [y, Y] \iff x \leq y$ and $X \leq Y$. As particular intervals of the weak order, the standard parabolic cosets are thus naturally ordered by restriction of the poset of intervals of the weak order: $xW_I \leq yW_J \iff x \leq y$ and $xw_{\circ,I} \leq yw_{\circ,J}$. We first give two equivalent descriptions of this order (see Section (2.2.3)): the first one, in terms of cover relations, was originally studied in (Krob et al., 2001; Palacios & Ronco, 2006) (see Section (2.2.1)), while the second one generalizes the characterization of the weak order in terms of inversion sets (see Section (2.2.2)). We then use these characterizations to prove that this poset is in fact a lattice (see Section (2.2.4)) and to study some of its order theoretic properties (see Section (2.2.5)).

2.2.1 Original definition by cover relations

We start from the original definition in terms of cover relations, which was given for the symmetric group by D. Krob, M. Latapy, J.-C. Novelli, H.-D. Phan and S. Schwer in (Krob et al., 2001), then extended for arbitrary finite Coxeter groups by P. Palacios and M. Ronco in (Palacios & Ronco, 2006).

Definition 2.2.1 ((Krob et al., 2001; Palacios & Ronco, 2006)) The *(right) facial weak order* is the order $\leq$ on the Coxeter complex $\mathcal{P}_W$ defined by cover relations of two types:

$$(1) \qquad xW_I \lessdot xW_{I\cup\{s\}} \qquad \text{if } s \notin I \text{ and } x \in W^{I\cup\{s\}},$$

$$(2) \qquad xW_I \lessdot xw_{\circ,I}w_{\circ,I\smallsetminus\{s\}}W_{I\smallsetminus\{s\}} \qquad \text{if } s \in I,$$

where $I \subseteq S$ and $x \in W^I$.

We have illustrated the facial weak order on the faces of the permutahedron in types A_2 and B_2 in Figure (2.4) and in type A_3 in Figure (2.5).

Remark 2.2.2 (1) These cover relations translate to the following geometric conditions on faces of the permutahedron $\mathsf{Perm}(W)$: a face F is covered by a face G if and only if either F is a facet of G with the same weak order minimum, or G is a facet of F with the same weak order maximum.

(2) Consider the natural inclusion $x \mapsto xW_\varnothing$ from W to $\mathcal{P}_W$. For $x \lessdot xs$ in weak order, we have $xW_\varnothing \lessdot xW_{\{s\}} \lessdot xsW_\varnothing$ in facial weak order. By transitivity, all relations in the classical weak order are thus relations in the facial weak order. Although it is not obvious at first sight from Definition (2.2.1), we will see in Corollary (2.2.17) that the restriction of the facial weak order to the vertices of $\mathcal{P}_W$ precisely coincides with the weak order.

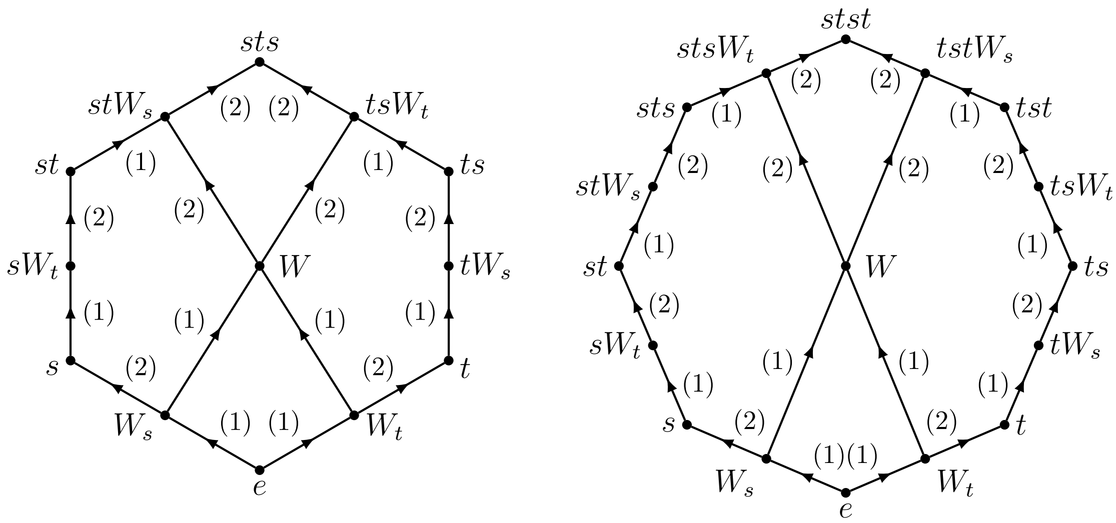


Figure 2.4: The facial weak order on the standard parabolic cosets of the Coxeter group of types A_2 and B_2. Edges are labelled with the cover relations of type (1) or (2) as in Definition (2.2.1).

(3) It is known that for $I \subseteq S$ the set of minimal length coset representatives W^I has a maximal length element $w_\circ w_{\circ,I}$. The element $w_{\circ,I} w_{\circ,I\smallsetminus\{s\}}$ is therefore the maximal length element of the set $W_I^{I\smallsetminus\{s\}} = W_I \cap W^{I\smallsetminus\{s\}}$, which is the set of minimal coset representatives of the cosets $W_I/W_{I\smallsetminus\{s\}}$, see (Geck & Pfeiffer, 2000, Section 2.2) for more details.

Example 2.2.3 As already mentioned, the facial weak order was first considered by D. Krob, M. Latapy, J.-C. Novelli, H.-D. Phan and S. Schwer (Krob et al., 2001) in type A. The standard parabolic cosets in type A_{n-1} correspond to ordered partitions of $[n]$, see Example (2.1.1). The weak order on ordered partitions of $[n]$ is the transitive closure of the cover relations

$$(1) \qquad (\lambda_1|\cdots|\lambda_i|\lambda_{i+1}|\cdots|\lambda_k) \lessdot (\lambda_1|\cdots|\lambda_i\lambda_{i+1}|\cdots|\lambda_k) \qquad \text{if } \lambda_i \ll \lambda_{i+1},$$

$$(2) \qquad (\lambda_1|\cdots|\lambda_i\lambda_{i+1}|\cdots|\lambda_k) \lessdot (\lambda_1|\cdots|\lambda_i|\lambda_{i+1}|\cdots|\lambda_k) \qquad \text{if } \lambda_{i+1} \ll \lambda_i,$$

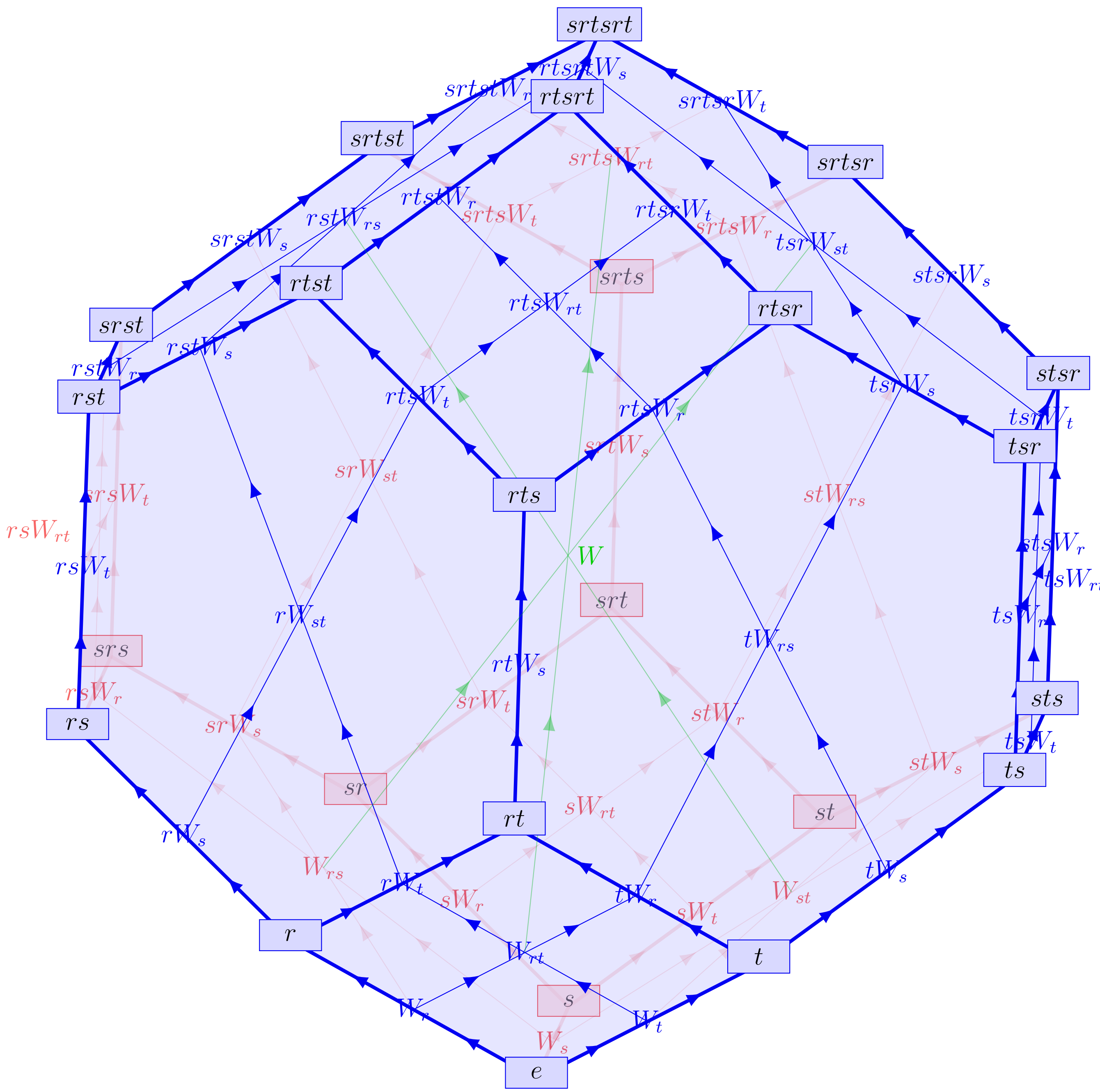


Figure 2.5: The facial weak order on the standard parabolic cosets of the Coxeter group of type A_3.

where the notation $X \ll Y$ is defined for $X, Y \subseteq \mathbb{N}$ by

$$X \ll Y \iff \max(X) < \min(Y) \iff x < y \text{ for all } x \in X \text{ and } y \in Y.$$

2.2.2 Root and weight inversion sets of standard parabolic cosets

We now define a collection of roots and a collection of weights associated to each standard parabolic coset. The notion of root inversion sets of standard parabolic cosets generalizes the inversion sets of elements of W (see Proposition (2.2.10)). We will use root inversion sets extensively for our study of the facial weak order. In contrast, weight inversion sets are not essential for our study of the facial weak order but will be relevant when we study its lattice congruences. We define them here as they are polar to the root inversion sets and appear naturally in our geometric intuition of the W-Coxeter arrangement and of the W-permutahedron (see Proposition (2.2.7)).

Definition 2.2.4 The *root inversion set* $\mathbf{R}(xW_I)$ and *weight inversion set* $\mathbf{W}(xW_I)$ of a standard parabolic coset xW_I are respectively defined by

$$\mathbf{R}(xW_I) := x\Big(\Phi^- \cup \Phi_I^+\Big) \subseteq \Phi \qquad \text{and} \qquad \mathbf{W}(xW_I) := x\Big(\nabla_{S\smallsetminus I}\Big) \subseteq \Omega$$

where $\nabla_{S\smallsetminus I}$ is the set of fundamental weights associated to the parabolic subgroup $W_{S\smallsetminus I}$.

Remark 2.2.5 Root inversion sets are known as "parabolic subsets of roots" in the sense of (Bourbaki, 1968, Section 1.7). In particular for any $x \in W$, the stabilizer of $\mathbf{R}(xW_I)$ for the action of W on the subsets of Φ is the parabolic subgroup xW_Ix^{-1}.

Example 2.2.6 Consider the facial weak order on the Coxeter group of type A_{n-1}, see Examples (2.1.1) and (2.2.3). Following (Krob et al., 2001), we define the *in-*

version table $\mathrm{inv}(\lambda) \in \{-1,0,1\}^{\binom{n}{2}}$ of an ordered partition λ of $[n]$ by

$$\mathrm{inv}(\lambda)_{i,j} = \begin{cases} -1 & \text{if } \lambda^{-1}(i) < \lambda^{-1}(j), \\ 0 & \text{if } \lambda^{-1}(i) = \lambda^{-1}(j), \\ 1 & \text{if } \lambda^{-1}(i) > \lambda^{-1}(j). \end{cases}$$

The root inversion set of a parabolic coset xW_I of $\mathfrak{S}_n$ is encoded by the inversion table of the corresponding ordered partition λ. We have

$$\mathrm{inv}(\lambda)_{i,j} = \begin{cases} -1 & \text{if } e_i - e_j \in \mathbf{R}(xW_I) \text{ but } e_j - e_i \notin \mathbf{R}(xW_I), \\ 0 & \text{if } e_i - e_j \in \mathbf{R}(xW_I) \text{ and } e_j - e_i \in \mathbf{R}(xW_I), \\ 1 & \text{if } e_i - e_j \notin \mathbf{R}(xW_I) \text{ but } e_j - e_i \in \mathbf{R}(xW_I). \end{cases}$$

The following statement gives the precise connection to the geometry of the W-permutahedron and is illustrated on Figure (2.6) for the Coxeter group of type A_2.

Proposition 2.2.7 *Let xW_I be a standard parabolic coset of W. Then*

(i) $\mathrm{cone}(\mathbf{R}(xW_I))$ *is the inner primal cone of the face* $\mathbf{F}(xW_I)$ *of* $\mathsf{Perm}(W)$*,*

(ii) $\mathrm{cone}(\mathbf{W}(xW_I))$ *is the outer normal cone of the face* $\mathbf{F}(xW_I)$ *of* $\mathsf{Perm}(W)$*,*

(iii) the cones generated by the root inversion set and by the weight inversion set of xW_I are polar to each other:

$$\mathrm{cone}(\mathbf{R}(xW_I))^\diamond = \mathrm{cone}(\mathbf{W}(xW_I)).$$

Proof. On the one hand, the inner primal cone of $\mathbf{F}(W_I)$ is generated by the vectors $\Phi^- \cup \Phi_I^+ = \mathbf{R}(eW_I)$. On the other hand, the outer normal cone of $\mathbf{F}(W_I)$ is generated by the normal vectors of $\mathbf{F}(W_I)$, *i.e.*, by $\nabla_{S \smallsetminus I} = \mathbf{W}(eW_I)$. The first two points then follow by applying the orthogonal transformation x and the last point is an immediate consequence of the first two. $\square$

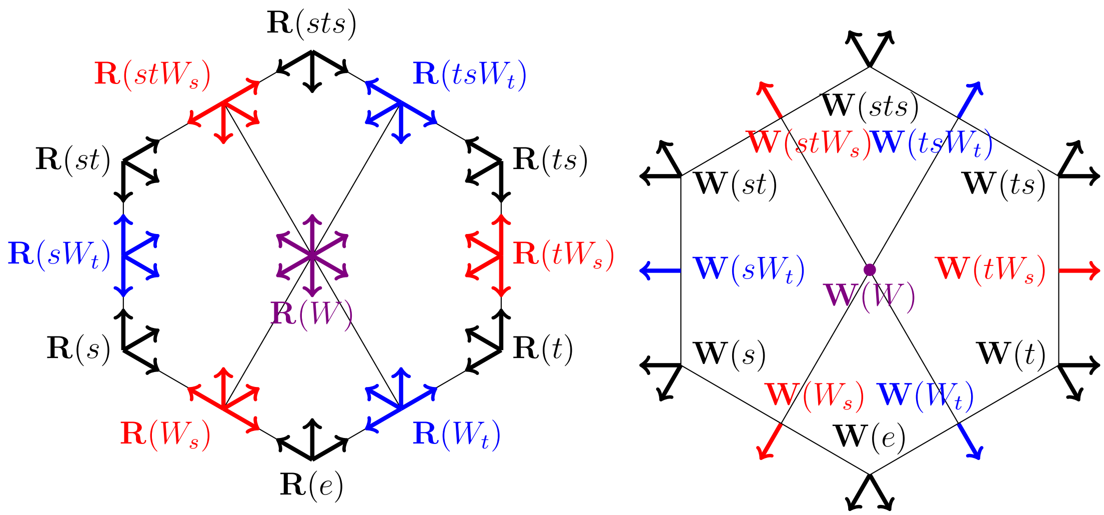


Figure 2.6: The root inversion sets (left) and weight inversion sets (right) of the standard parabolic cosets in type A_2. Note that positive roots point downwards.

It is well-known that the map $\mathbf{N}$, sending an element $w \in W$ to its inversion set $\mathbf{N}(w) = \Phi^+ \cap w(\Phi^-)$ is injective, see for instance (Hohlweg & Labbé, 2016, Section 2). The following corollary is the analogue for the maps $\mathbf{R}$ and $\mathbf{W}$.

Corollary 2.2.8 *The maps* $\mathbf{R}$ *and* $\mathbf{W}$ *are both injective.*

Proof. A face of a polytope is characterized by its inner primal cone (resp. outer normal cone). □

In a finite Coxeter group, a subset R of Φ^+ is an inversion set if and only if it is separable from its complement by a linear hyperplane, or equivalently if and only if both R and its complement $\Phi^+ \smallsetminus R$ are convex (meaning that $R = \Phi^+ \cap \operatorname{cone}(R)$). The following statement gives an analogue for root inversion sets.

Corollary 2.2.9 *The following assertions are equivalent for a subset* R *of* Φ*:*

(i) $R = \mathbf{R}(xW_I)$ *for some coset* $xW_I \in \mathcal{P}_W$*,*

(ii) $R = \{\alpha \in \Phi \mid \psi(\alpha) \geq 0\}$ *for some linear function* $\psi : V \to \mathbb{R}$*,*

(iii) $R = \Phi \cap \operatorname{cone}(R)$ *and* $R \cap \{\pm\alpha\} \neq \varnothing$ *for all* $\alpha \in \Phi$*.*

Proof. According to Proposition (2.2.7), for any coset xW_I, the set $\mathbf{R}(xW_I)$ is the set of roots in the inner normal cone of the face $\mathbf{F}(xW_I)$ of $\mathsf{Perm}(W)$. For any linear function $\psi : V \to \mathbb{R}$, the set $\{\alpha \in \Phi \mid \psi(\alpha) \geq 0\}$ is the set of roots in the inner normal cone of the face of $\mathsf{Perm}(W)$ defined by ψ. Since any face is defined by at least one linear function and any linear function defines a face, we get (i) $\Longleftrightarrow$ (ii). The equivalence (ii) $\Longleftrightarrow$ (iii) is immediate. □

Our next three statements concern the root inversion set $\mathbf{R}(xW_\varnothing)$ for $x \in W$. For brevity we write $\mathbf{R}(x)$ instead of $\mathbf{R}(xW_\varnothing)$. We first connect the root inversion set $\mathbf{R}(x)$ to the inversion set $\mathbf{N}(x)$, to reduced words for x, and to the root inversion sets $\mathbf{R}(xw_\circ)$ and $\mathbf{R}(w_\circ x)$.

Proposition 2.2.10 *For any* $x \in W$*, the root inversion set* $\mathbf{R}(x)$ *has the following properties.*

(i) $\mathbf{R}(x) = \mathbf{N}(x) \cup -\Big(\Phi^+ \smallsetminus \mathbf{N}(x)\Big)$ *where* $\mathbf{N}(x) = \Phi^+ \cap x(\Phi^-)$ *is the (left) inversion set of* x*. In other words,*

$$\mathbf{R}(x) \cap \Phi^+ = \mathbf{N}(x) \qquad \textit{and} \qquad \mathbf{R}(x) \cap \Phi^- = -\Big(\Phi^+ \smallsetminus \mathbf{N}(x)\Big).$$

(ii) *If* $x = s_1 s_2 \cdots s_k$ *is reduced, then*

$$\mathbf{R}(x) = \Phi^- \triangle \{\pm\alpha_{s_1}, \pm s_1(\alpha_{s_2}), \ldots, \pm s_1 \cdots s_{k-1}(\alpha_{s_k})\}.$$

(iii) $\mathbf{R}(xw_\circ) = -\mathbf{R}(x)$ *and* $\mathbf{R}(w_\circ x) = w_\circ\Big(\mathbf{R}(x)\Big)$*.*

Proof. For (i) we observe that $\mathbf{R}(x) = x(\Phi^-) = \Big(\Phi^+ \cap x(\Phi^-)\Big) \cup \Big(\Phi^- \cap x(\Phi^-)\Big)$. By definition of the inversion set we get

$$\mathbf{R}(x) = \mathbf{N}(x) \cup -\Big(\Phi^+ \cap x(\Phi^+)\Big) = \mathbf{N}(x) \cup -\Big(\Phi^+ \smallsetminus \mathbf{N}(x)\Big).$$

(ii) then follows from the fact that $\mathbf{N}(x) = \{\alpha_{s_1}, s_1(\alpha_{s_2}), \dots, s_1 \cdots s_{k-1}(\alpha_{s_k})\}$. Finally, (iii) follows from the definition of $\mathbf{R}$ and the fact that $w_\circ(\Phi^+) = \Phi^-$. □

The next statement gives a characterization of the (classical) weak order in terms of root inversion sets, which generalizes the characterization of the weak order in term of inversion sets. We will see later in Theorem (2.2.14) that the same characterization holds for the facial weak order.

Corollary 2.2.11 *For $x, y \in W$, we have*

$$\begin{aligned} x \le y &\iff \mathbf{R}(x) \smallsetminus \mathbf{R}(y) \subseteq \Phi^- \quad \textit{and} \quad \mathbf{R}(y) \smallsetminus \mathbf{R}(x) \subseteq \Phi^+, \\ &\iff \mathbf{R}(x) \cap \Phi^+ \subseteq \mathbf{R}(y) \cap \Phi^+ \quad \textit{and} \quad \mathbf{R}(x) \cap \Phi^- \supseteq \mathbf{R}(y) \cap \Phi^-. \end{aligned}$$

Proof. We observe from Proposition (2.2.10) (i) that

$$\mathbf{R}(x) \smallsetminus \mathbf{R}(y) = \Big(\mathbf{N}(x) \smallsetminus \mathbf{N}(y)\Big) \cup -\Big(\mathbf{N}(y) \smallsetminus \mathbf{N}(x)\Big).$$

The result thus follows immediately from the fact that $x \le y \iff \mathbf{N}(x) \subseteq \mathbf{N}(y)$, see Section (2.1.3). □

Finally, we observe that the root and weight inversion sets of a parabolic coset xW_I can be computed from that of its minimal and maximal length representatives x and $xw_{\circ,I}$.

Proposition 2.2.12 *The root and weight inversion sets of xW_I can be computed from those of x and $xw_{\circ,I}$ by*

$$\mathbf{R}(xW_I) = \mathbf{R}(x) \cup \mathbf{R}(xw_{\circ,I}) \qquad \textit{and} \qquad \mathbf{W}(xW_I) = \mathbf{W}(x) \cap \mathbf{W}(xw_{\circ,I}).$$

Proof. For the root inversion set, we just write

$$\begin{aligned} \mathbf{R}(x) \cup \mathbf{R}(xw_{\circ,I}) &= x(\Phi^-) \cup xw_{\circ,I}(\Phi^-) = x(\Phi^-) \cup x(\Phi^- \bigtriangleup \Phi_I) \\ &= x(\Phi^- \cup \Phi_I^+) = \mathbf{R}(xW_I). \end{aligned}$$

The proof is similar for the weight inversion set (or can be derived from Proposition (2.2.7)). □

Corollary 2.2.13 *For any coset xW_I, we have*

$$\mathbf{R}(xW_I) \cap \Phi^- = \mathbf{R}(x) \cap \Phi^- \qquad \textit{and} \qquad \mathbf{R}(xW_I) \cap \Phi^+ = \mathbf{R}(xw_{\circ,I}) \cap \Phi^+$$

Proof. Since $x \leq xw_{\circ,I}$, Corollary (2.2.11) ensures that $\mathbf{R}(x) \cap \Phi^+ \subseteq \mathbf{R}(xw_{\circ,I}) \cap \Phi^+$ and $\mathbf{R}(x) \cap \Phi^- \supseteq \mathbf{R}(xw_{\circ,I}) \cap \Phi^-$. Therefore, we obtain from Proposition (2.2.12) that

$$\mathbf{R}(xW_I) \cap \Phi^- = \Big(\mathbf{R}(x) \cup \mathbf{R}(xw_{\circ,I})\Big) \cap \Phi^- = \mathbf{R}(x) \cap \Phi^-,$$

and similarly

$$\mathbf{R}(xW_I) \cap \Phi^+ = \Big(\mathbf{R}(x) \cup \mathbf{R}(xw_{\circ,I})\Big) \cap \Phi^+ = \mathbf{R}(xw_{\circ,I}) \cap \Phi^+.$$

□

2.2.3 Three equivalent characterizations of the facial weak order

We are now ready to give three equivalent characterizations of the facial weak order: the original one in terms of cover relations (Palacios & Ronco, 2006), the geometric one in terms of root inversion sets, and the combinatorial one as the subposet of the poset of intervals of the weak order induced by the standard parabolic subgroups. Using the root inversion sets defined in the previous section, we now give two equivalent characterizations of the facial weak order defined by P. Palacios and M. Ronco in (Palacios & Ronco, 2006) (see Definition (2.2.1)). In type A, the equivalence (i) $\iff$ (ii) below is stated in (Krob et al., 2001, Theorem 5) in terms of half-inversion tables (see Examples (2.2.3) and (2.2.6)).

Theorem 2.2.14 *The following conditions are equivalent for two standard parabolic cosets xW_I and yW_J in $\mathcal{P}_W$:*

(i) $xW_I \leq yW_J$ *in facial weak order,*

(ii) $\mathbf{R}(xW_I) \smallsetminus \mathbf{R}(yW_J) \subseteq \Phi^-$ *and* $\mathbf{R}(yW_J) \smallsetminus \mathbf{R}(xW_I) \subseteq \Phi^+$*,*

(iii) $x \leq y$ *and* $xw_{\circ,I} \leq yw_{\circ,J}$ *in weak order.*

Proof. We will prove that (i)⇒(iii)⇒(ii)⇒(i), the last implication being the most technical.

The implication (i)⇒(iii) is immediate. The first cover relation keeps x and transforms $xw_{\circ,I}$ to $xw_{\circ,I\cup\{s\}}$, and the second cover relation transforms x to $xw_{\circ,I}w_{\circ,I\smallsetminus\{s\}}$ but keeps $xw_{\circ,I}$. Since $xw_{\circ,I} \leq xw_{\circ,I\cup\{s\}}$ and $x \leq xw_{\circ,I}w_{\circ,I\smallsetminus\{s\}}$, we obtain the result by transitivity.

For the implication (iii)⇒(ii), Corollary (2.2.11) ensures that $\mathbf{R}(x) \smallsetminus \mathbf{R}(y) \subseteq \Phi^-$ and $\mathbf{R}(y) \smallsetminus \mathbf{R}(x) \subseteq \Phi^+$ since $x \leq y$, and similarly that $\mathbf{R}(xw_{\circ,I}) \smallsetminus \mathbf{R}(yw_{\circ,J}) \subseteq \Phi^-$ and $\mathbf{R}(yw_{\circ,J}) \smallsetminus \mathbf{R}(xw_{\circ,I}) \subseteq \Phi^+$ since $xw_{\circ,I} \leq yw_{\circ,J}$. From Proposition (2.2.12), we therefore obtain

$$\begin{aligned}\mathbf{R}(xW_I) \smallsetminus \mathbf{R}(yW_J) &= \Big(\mathbf{R}(x) \cup \mathbf{R}(xw_{\circ,I})\Big) \smallsetminus \Big(\mathbf{R}(y) \cup \mathbf{R}(yw_{\circ,J})\Big) \\ &\subseteq \Big(\mathbf{R}(x) \smallsetminus \mathbf{R}(y)\Big) \cup \Big(\mathbf{R}(xw_{\circ,I}) \smallsetminus \mathbf{R}(yw_{\circ,J})\Big) \\ &\subseteq \Phi^-.\end{aligned}$$

We prove similarly that $\mathbf{R}(yW_J) \smallsetminus \mathbf{R}(xW_I) \subseteq \Phi^+$.

We now focus on the implication (ii)⇒(i). We consider two standard parabolic cosets xW_I and yW_J which satisfy Condition (ii) and construct a path of cover relations as in Definition (2.2.1) between them. We proceed by induction on the cardinality $|\,\mathbf{R}(xW_I) \triangle \mathbf{R}(yW_J)|$.

First, if $|\,\mathbf{R}(xW_I) \triangle \mathbf{R}(yW_J)| = 0$, then $\mathbf{R}(xW_I) = \mathbf{R}(yW_J)$, which ensures that $xW_I = yW_J$ by Corollary (2.2.8). Assume now that $|\,\mathbf{R}(xW_I) \triangle \mathbf{R}(yW_J)| > 0$.

So we either have $\mathbf{R}(xW_I) \smallsetminus \mathbf{R}(yW_J) \neq \varnothing$ or $\mathbf{R}(yW_J) \smallsetminus \mathbf{R}(xW_I) \neq \varnothing$. We consider only the case $\mathbf{R}(xW_I) \smallsetminus \mathbf{R}(yW_J) \neq \varnothing$, the other case being symmetric.

To proceed by induction, our goal is to find a new coset zW_K so that

- $xW_I \lessdot zW_K$ is one of the cover relations of Definition (2.2.1),
- zW_K and yW_J still satisfy Condition (ii), and
- $\mathbf{R}(zW_K) \triangle \mathbf{R}(yW_J) \subsetneq \mathbf{R}(xW_I) \triangle \mathbf{R}(yW_J)$.

Indeed, by induction hypothesis, there will exist a path from zW_K to yW_J consisting of cover relations as in Definition (2.2.1). Adding the first step $xW_I \lessdot zW_K$, we then obtain a path from xW_I to yW_J.

To construct this new coset zW_K and its root inversion set $\mathbf{R}(zW_K)$, we will add or delete (at least) one root from $\mathbf{R}(xW_I)$. We first claim that there exists $s \in S$ such that $-x(\alpha_s) \notin \mathbf{R}(yW_J)$. Otherwise, we would have $x(-\Delta) \subseteq \mathbf{R}(yW_J)$. Since $\Phi^- = \mathrm{cone}(-\Delta) \cap \Phi$ and $\mathbf{R}(yW_J) = \mathrm{cone}(\mathbf{R}(yW_J)) \cap \Phi$, this would imply that $x(\Phi^-) \subseteq \mathbf{R}(yW_J)$. Moreover, $x(\Phi_I^+) \subseteq \Phi^+$ since $x \in W^I$. Thus we would obtain

$$\begin{aligned}\mathbf{R}(xW_I) \smallsetminus \mathbf{R}(yW_J) &= \big(x(\Phi^-) \cup x(\Phi_I^+)\big) \smallsetminus \mathbf{R}(yW_J) \\ &\subseteq \big(x(\Phi^-) \smallsetminus \mathbf{R}(yW_J)\big) \cup x(\Phi_I^+) \\ &\subseteq \Phi^+.\end{aligned}$$

However, $\mathbf{R}(xW_I) \smallsetminus \mathbf{R}(yW_J) \subseteq \Phi^-$ by Condition (ii). Hence we would obtain that $\mathbf{R}(xW_I) \smallsetminus \mathbf{R}(yW_J) \subseteq \Phi^+ \cap \Phi^- = \varnothing$, contradicting our assumption.

For the remaining of the proof we fix $s \in S$ such that $-x(\alpha_s) \notin \mathbf{R}(yW_J)$ and we set $\beta := x(\alpha_s)$. By definition, we have $-\beta \in \mathbf{R}(xW_I) \smallsetminus \mathbf{R}(yW_J)$. Moreover, since $-\beta \notin \mathbf{R}(yW_J)$ and $\mathbf{R}(yW_J) \cup -\mathbf{R}(yW_J) \supseteq y(\Phi^-) \cup -y(\Phi^-) = \Phi$, we have $\beta \in \mathbf{R}(yW_J)$. We now distinguish two cases according to whether or not $\beta \in \mathbf{R}(xW_I)$, that is, on whether or not $s \in I$. In both cases, we will need the following observation: since $\mathbf{R}(xW_I) \smallsetminus \mathbf{R}(yW_J) \subseteq \Phi^-$, we have

$$x(\Phi_I^+) \subseteq \Phi^+ \cap \mathbf{R}(xW_I) \subseteq \mathbf{R}(yW_J). \qquad (\star)$$

First case: $s \notin I$**.** Since $-x(\alpha_s) = -\beta \in \mathbf{R}(xW_I) \smallsetminus \mathbf{R}(yW_J) \subseteq \Phi^-$, we have that $x(\alpha_s) \in \Phi^+$ and thus $x \in W^{I\cup\{s\}}$. We can therefore consider the standard parabolic coset $zW_K := xW_{I\cup\{s\}}$ where $z := x$ and $K := I \cup \{s\}$. Its root inversion set is given by $\mathbf{R}(zW_K) = \mathbf{R}(xW_I) \cup x(\Phi_K^+)$. Note that $xW_I \lessdot zW_K$ is a cover relation of type (1) in Definition (2.2.1). It thus remains to show that zW_K and yW_J still satisfy Condition (ii) and that $\mathbf{R}(zW_K) \triangle \mathbf{R}(yW_J) \subsetneq \mathbf{R}(xW_I) \triangle \mathbf{R}(yW_J)$.

Since $\beta = x(\alpha_s) \in \mathbf{R}(yW_J)$ and using Observation $(\star)$ above, we thus have

$$x(\Phi_K^+) = \operatorname{cone}\Big(\{\beta\} \cup x(\Phi_I^+)\Big) \cap \Phi \subseteq \mathbf{R}(yW_J).$$

Therefore we obtain

$$\mathbf{R}(zW_K) \smallsetminus \mathbf{R}(yW_J) = \mathbf{R}(xW_I) \cup x(\Phi_K^+) \smallsetminus \mathbf{R}(yW_J) \subseteq \mathbf{R}(xW_I) \smallsetminus \mathbf{R}(yW_J) \subseteq \Phi^-.$$

Moreover, since $\mathbf{R}(xW_I) \subseteq \mathbf{R}(zW_K)$,

$$\mathbf{R}(yW_J) \smallsetminus \mathbf{R}(zW_K) \subseteq \mathbf{R}(yW_J) \smallsetminus \mathbf{R}(xW_I) \subseteq \Phi^+.$$

Therefore, we proved that the cosets zW_K and yW_J still satisfy Condition (ii) and that $\mathbf{R}(zW_K) \triangle \mathbf{R}(yW_J) \subseteq \mathbf{R}(xW_I) \triangle \mathbf{R}(yW_J)$. The strict inclusion then follows since $-\beta$ belongs to $\mathbf{R}(xW_I) \triangle \mathbf{R}(yW_J)$ but not to $\mathbf{R}(zW_K) \triangle \mathbf{R}(yW_J)$.

Second case: $s \in I$**.** Let $s^\star := w_{\circ,I} s w_{\circ,I}$. Consider the standard parabolic coset zW_K where $K := I \smallsetminus \{s^\star\}$ and $z := xw_{\circ,I}w_{\circ,K}$. Note that $xW_I \lessdot zW_K$ is a cover relation of type (2) in Definition (2.2.1). It thus remains to show that zW_K and yW_J still satisfy Condition (ii) and that $\mathbf{R}(zW_K) \triangle \mathbf{R}(yW_J) \subsetneq \mathbf{R}(xW_I) \triangle \mathbf{R}(yW_J)$.

We first prove that

$$\mathbf{R}(xW_I) = \mathbf{R}(zW_K) \cup x(\Phi_I^-) \smallsetminus x(\Phi^-_{I \smallsetminus \{s\}}). \qquad (\clubsuit)$$

Observe that $w_{\circ,I}(\Phi^-) = \Phi^- \triangle \Phi_I$ and that $w_{\circ,I}(\Phi_K) = w_{\circ,I}(\Phi_{I \smallsetminus \{s^\star\}}) = \Phi_{I \smallsetminus \{s\}}$. Therefore,

$$\begin{aligned} & w_{\circ,I} w_{\circ,K}(\Phi^-) = w_{\circ,I}(\Phi^- \triangle \Phi_K) = \Phi^- \triangle (\Phi_I \smallsetminus \Phi_{I \smallsetminus \{s\}}) \\ \text{and} \quad & w_{\circ,I} w_{\circ,K}(\Phi_K^+) = w_{\circ,I}(\Phi_K^-) = \Phi^+_{I \smallsetminus \{s\}}. \end{aligned}$$

Therefore we obtain the desired equality:

$$\begin{aligned} \mathbf{R}(xW_I) &= x(\Phi^- \cup \Phi_I^+) \\ &= x\Big(\Phi^- \triangle (\Phi_I \smallsetminus \Phi_{I \smallsetminus \{s\}})\Big) \cup x(\Phi^+_{I \smallsetminus \{s\}}) \cup x(\Phi_I^-) \smallsetminus x(\Phi^-_{I \smallsetminus \{s\}}) \\ &= x w_{\circ,I} w_{\circ,K}(\Phi^-) \cup x w_{\circ,I} w_{\circ,K}(\Phi_K^+) \cup x(\Phi_I^-) \smallsetminus x(\Phi^-_{I \smallsetminus \{s\}}) \\ &= \mathbf{R}(zW_K) \cup x(\Phi_I^-) \smallsetminus x(\Phi^-_{I \smallsetminus \{s\}}). \end{aligned}$$

We now check that

$$\Big(x(\Phi_I^-) \smallsetminus x(\Phi^-_{I \smallsetminus \{s\}})\Big) \cap \mathbf{R}(yW_J) = \varnothing. \qquad (\spadesuit)$$

Indeed, assume that this set contains an element δ. We have $\delta = a(-\beta) + \gamma$, where $a > 0$ and $\gamma \in x(\Phi^-_{I \smallsetminus \{s\}})$. Therefore, $-\beta = (\delta - \gamma)/a$. Since $\delta \in \mathbf{R}(yW_J)$ and $-\gamma \in x(\Phi^+_{I \smallsetminus \{s\}}) \subseteq x(\Phi_I^+) \subseteq \mathbf{R}(yW_J)$ by Observation ($\star$) above, we would obtain that $-\beta \in \mathbf{R}(yW_J)$, contradicting our definition of β.

Now, combining Equations ($\clubsuit$) and ($\spadesuit$), we obtain

$$\mathbf{R}(yW_J) \smallsetminus \mathbf{R}(zW_K) \subseteq \mathbf{R}(yW_J) \smallsetminus \mathbf{R}(xW_I) \subseteq \Phi^+.$$

Moreover, since $\mathbf{R}(zW_K) \subseteq \mathbf{R}(xW_I)$,

$$\mathbf{R}(zW_K) \smallsetminus \mathbf{R}(yW_J) \subseteq \mathbf{R}(xW_I) \smallsetminus \mathbf{R}(yW_J) \subseteq \Phi^-.$$

Therefore, we proved that the cosets zW_K and yW_J still satisfy Condition (ii) and that $\mathbf{R}(zW_K) \triangle \mathbf{R}(yW_J) \subseteq \mathbf{R}(xW_I) \triangle \mathbf{R}(yW_J)$. The strict inclusion then follows as $-\beta$ belongs to $\mathbf{R}(xW_I) \triangle \mathbf{R}(yW_J)$ but not to $\mathbf{R}(zW_K) \triangle \mathbf{R}(yW_J)$. This concludes the proof. $\square$

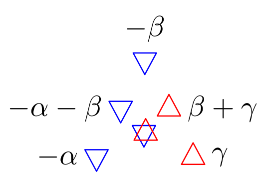
Remark 2.2.15 Observe that our characterization of the facial weak order in terms of root inversion sets given in Theorem (2.2.14) (ii) is equivalent to the following: $xW_I \leq yW_J$ if and only if

(ii') $\mathbf{R}(xW_I) \cap \Phi^+ \subseteq \mathbf{R}(yW_J) \cap \Phi^+$ and $\mathbf{R}(xW_I) \cap \Phi^- \supseteq \mathbf{R}(yW_J) \cap \Phi^-$.

Example 2.2.16 We have illustrated the facial weak order by the means of root inversion sets in Figure (2.7). In this figure each face is labelled by its root inversion set. To visualize the roots, we consider the affine plane P passing through the simple roots $\{\alpha, \beta, \gamma\}$. A positive (resp. negative) root ρ is then seen as a red upward (resp. blue downward) triangle placed at the intersection of $\mathbb{R}\rho$ with the plane P. For instance,

$$\mathbf{R}(cbW_a) = \{\gamma, \beta + \gamma, \alpha + \beta + \gamma\} \cup \{-\alpha, -\beta, -\alpha - \beta - \gamma, -\alpha - \beta\}.$$

is labelled in Figure (2.7) by



Note that the star in the middle represents both $\alpha + \beta + \gamma$ and $-\alpha - \beta - \gamma$.

Using either our characterization of Theorem (2.2.14) (ii) together with Corollary (2.2.11), or our characterization of Theorem (2.2.14) (iii), we obtain that the facial weak order and the weak order coincide on the elements of W. Note that this is not at all obvious with the cover relations from Definition (2.2.1).

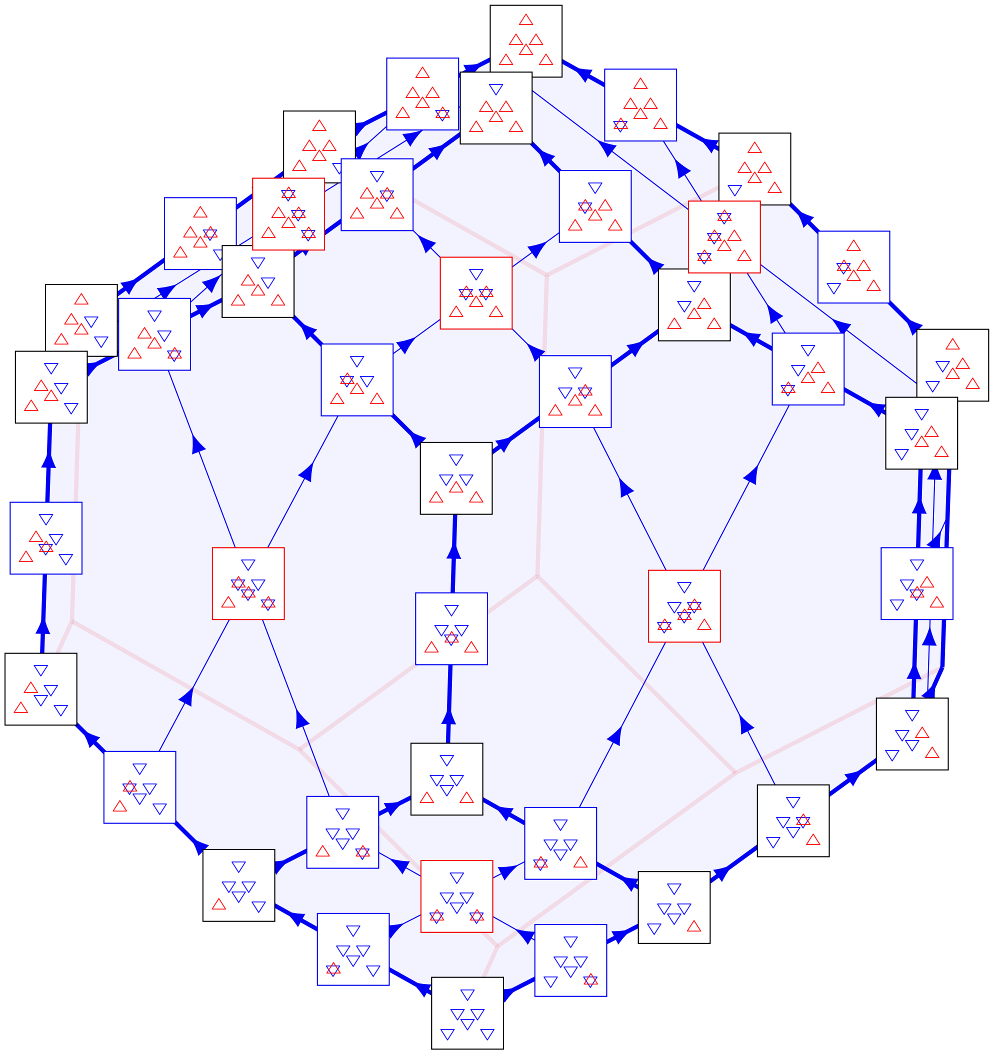


Figure 2.7: The facial weak order on the standard parabolic cosets of the Coxeter group of type A_3. Each coset xW_I is replaced by its root inversion set $\mathbf{R}(xW_I)$, represented as follows: down blue triangles stand for negative roots while up red triangles stand for positive roots, and the position of each triangle is given by the barycentric coordinates of the corresponding root with respect to the three simple roots (α on the bottom left, β on the top, and γ on the bottom right). See Example (2.2.16) for more details.

Corollary 2.2.17 *For any $x, y \in W$, we have $x \le y$ in weak order if and only if $xW_\varnothing \le yW_\varnothing$ in facial weak order.*

The weak order anti-automorphisms $x \mapsto xw_\circ$ and $x \mapsto w_\circ x$ and the automorphism $x \mapsto w_\circ x w_\circ$ correspond to maps on standard parabolic cosets. The following statement gives the precise definitions of the corresponding maps.

Proposition 2.2.18 *The maps*

$$xW_I \longmapsto w_\circ x w_{\circ,I} W_I \qquad \textit{and} \qquad xW_I \longmapsto x w_{\circ,I} w_\circ W_{w_\circ I w_\circ}$$

are anti-automorphisms of the facial weak order. Consequently, the map

$$xW_I \longmapsto w_\circ x w_\circ W_{w_\circ I w_\circ}$$

is an automorphism of the facial weak order.

Proof. Using the characterization of the facial weak order given in Theorem (2.2.14) (ii) we just need to observe that

$$\mathbf{R}(w_\circ x w_{\circ,I} W_I) = w_\circ\Big(\mathbf{R}(xW_I)\Big) \qquad \text{and} \qquad \mathbf{R}(x w_{\circ,I} w_\circ W_{w_\circ I w_\circ}) = -\,\mathbf{R}(xW_I).$$

This follows immediately from Propositions (2.2.12) and (2.2.10) (iii). □

2.2.4 The facial weak order is a lattice

In this section, we show that the facial weak order on standard parabolic cosets is a lattice. It generalizes the result for the symmetric group due to D. Krob, M. Latapy, J.-C. Novelli, H.-D. Phan and S. Schwer (Krob et al., 2001) to the facial weak order on arbitrary finite Coxeter groups introduced by P. Palacios and M. Ronco (Palacios & Ronco, 2006). It also gives a precise description of the meets and joins in this lattice. The characterizations of the facial weak order given in Theorem (2.2.14) are key here.

Theorem 2.2.19 *The facial weak order* $(\mathcal{P}_W, \leq)$ *is a lattice. The meet and join of two standard parabolic cosets* xW_I *and* yW_J *are given by:*

$$xW_I \wedge yW_J = z_\wedge W_{K_\wedge} \quad \textit{where} \quad z_\wedge = x \wedge y \quad \textit{and} \quad K_\wedge = D_L\Big(z_\wedge^{-1}(xw_{\circ,I} \wedge yw_{\circ,J})\Big),$$

$$xW_I \vee yW_J = z_\vee W_{K_\vee} \quad \textit{where} \quad z_\vee = xw_{\circ,I} \vee yw_{\circ,J} \quad \textit{and} \quad K_\vee = D_L\Big(z_\vee^{-1}(x \vee y)\Big).$$

Let us provide some intuition on this statement. Note that the poset of intervals of a lattice L is again a lattice whose meet and join are given by $[x, X] \wedge [y, Y] = [x \wedge y, X \wedge Y]$ and $[x, X] \vee [y, Y] = [x \vee y, X \vee Y]$. The interval $[x \wedge y, xw_{\circ,I} \wedge yw_{\circ,J}]$ is thus the meet interval of the two standard parabolic coset xW_I and yW_J in the lattice of intervals of the weak order. However, this meet interval is not anymore a standard parabolic coset. The meet $xW_I \wedge yW_J$ in the facial weak order is obtained as the biggest parabolic coset in this meet interval $[x \wedge y, xw_{\circ,I} \wedge yw_{\circ,J}]$ containing $x \wedge y$. Similarly, the join $xW_I \vee yW_J$ is the biggest parabolic coset in the join interval $[x \vee y, xw_{\circ,I} \vee yw_{\circ,J}]$ containing $xw_{\circ,I} \vee yw_{\circ,J}$.

Note that in the second point of Theorem (2.2.19), the minimal representative of the coset $z_\vee W_{K_\vee}$ is in fact $z_\vee w_{\circ,K_\vee}$, not $z_\vee$. Unlike in the rest of the paper, we take the liberty to use another coset representative than the minimal one to underline the symmetry between meet and join in the facial weak order.

Example 2.2.20 Before proving the above statement, we give two examples of computations of the meet and the join in the facial weak order.

(i) Consider first the Coxeter system

$$\Big\langle r, s, t \;\Big|\; r^2 = s^2 = t^2 = (rs)^3 = (st)^3 = (rt)^2 = 1 \Big\rangle$$

of type A_3. Figure (2.5) shows the facial weak order of $\mathcal{P}_W$ and is a good way to follow along. To find the meet of $tsrW_{st}$ and $rtsW_\varnothing$, we compute:

$$z_\wedge = tsr \wedge rts = t,$$

$$K_\wedge = D_L\Big(z_\wedge^{-1}(tsrw_{\circ,st} \wedge rtsw_{\circ,\varnothing})\Big) = D_L\Big(t(tsrsts \wedge rts)\Big) = D_L(t(rts)) = \{r\}.$$

Thus we have that $tsrW_{st} \wedge rtsW_{\varnothing} = z_{\wedge} W_{K_{\wedge}} = tW_r$.

(ii) For a slightly more complex example, consider the Coxeter system

$$\left\langle r, s, t \;\middle|\; r^2 = s^2 = t^2 = (rs)^4 = (st)^3 = (rt)^2 = 1 \right\rangle$$

of type B_3. To find the join of $rstW_{rs}$ and $tsrsW_{\varnothing}$, we compute:

$$z_{\vee} = rstw_{\circ,rs} \vee tsrsw_{\circ,\varnothing} = rstrsrs \vee tsrs = rtsrtsrt$$
$$K_{\vee} = D_L\Big(z_{\vee}^{-1}(rst \vee tsrs)\Big) = D_L\Big(trstrstr(rtsrtst)\Big) = D_L(r) = \{r\}$$

Thus we see that

$$rstW_{rs} \vee tsrsw_{\circ,\varnothing} = z_{\vee} w_{\circ,K_{\vee}} W_{K_{\vee}} = rtsrtsrt(r)W_r = rtsrtstW_r.$$

Proof of Theorem (2.2.19). Throughout the proof we use the characterization of the facial weak order given in Theorem (2.2.14) (iii):

$$xW_I \leq yW_J \iff x \leq y \text{ and } xw_{\circ,I} \leq yw_{\circ,J}.$$

We first prove the existence of the meet, then use Proposition (2.2.18) to deduce the existence and formula for the join.

Existence of meet. For any $s \in K_{\wedge}$, we have

$$\begin{aligned}
\ell(xw_{\circ,I} \wedge yw_{\circ,J}) - \ell(sz_{\wedge}^{-1}) &\leq \ell\Big(sz_{\wedge}^{-1}(xw_{\circ,I} \wedge yw_{\circ,J})\Big) \\
&= \ell\Big(z_{\wedge}^{-1}(xw_{\circ,I} \wedge yw_{\circ,J})\Big) - 1 \\
&= \ell(xw_{\circ,I} \wedge yw_{\circ,J}) - \ell(z_{\wedge}^{-1}) - 1.
\end{aligned}$$

Indeed, the first inequality holds in general (for reduced or non-reduced words). The first equality follows from $s \in K_{\wedge} = D_L(z_{\wedge}^{-1}(xw_{\circ,I} \wedge yw_{\circ,J}))$. The last equality holds since $z_{\wedge} = x \wedge y \leq xw_{\circ,I} \wedge yw_{\circ,J}$. We deduce from this inequality that $\ell(z_{\wedge}) < \ell(z_{\wedge}s)$. Therefore, we have $z_{\wedge} \in W^{K_{\wedge}}$.

Since $K_\wedge = D_L(z_\wedge^{-1}(xw_{\circ,I} \wedge yw_{\circ,J}))$, we have $w_{\circ,K_\wedge} \leq z_\wedge^{-1}(xw_{\circ,I} \wedge yw_{\circ,J})$. Therefore $z_\wedge w_{\circ,K_\wedge} \leq xw_{\circ,I} \wedge yw_{\circ,J}$, since $z_\wedge \in W^{K_\wedge}$. We thus have $z_\wedge = x \wedge y \leq x$ and $z_\wedge w_{\circ,K_\wedge} \leq xw_{\circ,I} \wedge yw_{\circ,J} \leq xw_{\circ,I}$, which implies $z_\wedge W_{K_\wedge} \leq xW_I$, by Theorem (2.2.14) (ii). By symmetry, $z_\wedge W_{K_\wedge} \leq yW_J$.

It remains to show that $z_\wedge W_{K_\wedge}$ is the greatest lower bound. Consider a standard parabolic coset zW_K such that $zW_K \leq xW_I$ and $zW_K \leq yW_J$. We want to show that $zW_K \leq z_\wedge W_{K_\wedge}$, that is, $z \leq z_\wedge$ and $zw_{\circ,K} \leq z_\wedge w_{\circ,K_\wedge}$. The first inequality is immediate since $z \leq x$ and $z \leq y$ so that $z \leq x \wedge y = z_\wedge$. For the second one, we write the following reduced words: $x = zx'$, $y = zy'$, and $z_\wedge = zz'_\wedge$ where $z'_\wedge = x' \wedge y'$. Since $zw_{\circ,K} \leq xw_{\circ,I}$ and $zw_{\circ,K} \leq yw_{\circ,J}$, we have

$$zw_{\circ,K} \leq xw_{\circ,I} \wedge yw_{\circ,J} = zx'w_{\circ,I} \wedge zy'w_{\circ,J} = z(x'w_{\circ,I} \wedge y'w_{\circ,J}).$$

Thus $w_{\circ,K} \leq x'w_{\circ,I} \wedge y'w_{\circ,J}$, since all words are reduced here. Therefore

$$K \subseteq D_L(x'w_{\circ,I} \wedge y'w_{\circ,J}).$$

We now claim that $D_L(x'w_{\circ,I} \wedge y'w_{\circ,J}) \subseteq D_L(z'_\wedge w_{\circ,K_\wedge})$. To see it, consider $s \in D_L(x'w_{\circ,I} \wedge y'w_{\circ,J})$ and assume by contradiction that $s \notin D_L(z'_\wedge w_{\circ,K_\wedge})$. Then s does not belong to $D_L(z'_\wedge)$, since the expression $z'_\wedge w_{\circ,K_\wedge}$ is reduced. By Deodhar's Lemma (see Section (2.1.4)) we obtain that either $sz'_\wedge \in W^{K_\wedge}$ or $sz'_\wedge = z'_\wedge t$ where

$$t \in D_L\Big(z_\wedge'^{-1}(x'w_{\circ,I} \wedge y'w_{\circ,J})\Big) = D_L\Big(z_\wedge^{-1}(xw_{\circ,I} \wedge yw_{\circ,J})\Big) = K_\wedge.$$

In the first case we obtain

$$1 + \ell(z'_\wedge) + \ell(w_{\circ,K_\wedge}) = \ell(sz'_\wedge w_{\circ,K_\wedge}) = \ell(z'_\wedge w_{\circ,K_\wedge}) - 1 = \ell(z'_\wedge) + \ell(w_{\circ,K_\wedge}) - 1$$

a contradiction. In the second case, we get

$$\begin{aligned} 1 + \ell(z'_\wedge) + \ell(w_{\circ,K_\wedge}) &= \ell(sz'_\wedge w_{\circ,K_\wedge}) = \ell(z'_\wedge t w_{\circ,K_\wedge}) \\ &= \ell(z'_\wedge) + \ell(tw_{\circ,K_\wedge}) = \ell(z'_\wedge) + \ell(w_{\circ,K_\wedge}) - 1, \end{aligned}$$

a contradiction again. This proves that $D_L(x'w_{\circ,I} \wedge y'w_{\circ,J}) \subseteq D_L(z'_{\wedge}w_{\circ,K_{\wedge}})$.

To conclude the proof, we deduce from $K \subseteq D_L(x'w_{\circ,I} \wedge y'w_{\circ,J}) \subseteq D_L(z'_{\wedge}w_{\circ,K_{\wedge}})$ that $w_{\circ,K} \leq z'_{\wedge}w_{\circ,K_{\wedge}}$, and finally that $zw_{\circ,K} \leq zz'_{\wedge}w_{\circ,K_{\wedge}} = z_{\wedge}w_{\circ,K_{\wedge}}$ since all expressions are reduced. Since $z \leq z_{\wedge}$ and $zw_{\circ,K} \leq z_{\wedge}w_{\circ,K_{\wedge}}$, we have $zW_K \leq z_{\wedge}W_{K_{\wedge}}$ so that $z_{\wedge}W_{K_{\wedge}}$ is indeed the greatest lower bound.

Existence of join. The existence of the join follows from the existence of meet and the anti-automorphism $\Psi : xW_I \longmapsto w_{\circ}xw_{\circ,I}W_I$ from Proposition (2.2.18). Using the fact that $w_{\circ}(w_{\circ}u \wedge w_{\circ}v) = u \vee v$, we get the formula

$$\begin{aligned}
xW_I \vee yW_J &= \Psi\Big(\Psi(xW_I) \wedge \Psi(yW_J)\Big) \\
&= \Psi(w_{\circ}xw_{\circ,I}W_I \wedge w_{\circ}yw_{\circ,J}W_J) \\
&= \Psi\Big((w_{\circ}xw_{\circ,I} \wedge w_{\circ}yw_{\circ,J})W_{D_L((w_{\circ}xw_{\circ,I}\wedge w_{\circ}yw_{\circ,J})^{-1}(w_{\circ}x\wedge w_{\circ}y))}\Big) \\
&= \Psi\Big((w_{\circ}xw_{\circ,I} \wedge w_{\circ}yw_{\circ,J})W_{K_{\vee}}\Big) \\
&= w_{\circ}(w_{\circ}xw_{\circ,I} \wedge w_{\circ}yw_{\circ,J})w_{\circ,K_{\vee}}W_{K_{\vee}} \\
&= z_{\vee}w_{\circ,K_{\vee}}W_{K_{\vee}}.
\end{aligned}$$

□

We already observed in Corollary (2.2.17) that the classical weak order is a subposet of the facial weak order. The formulas of Theorem (2.2.19) ensure that it is also a sublattice.

Corollary 2.2.21 *The classical weak order is a sublattice of the facial weak order.*

Proof. If $I = J = \varnothing$, then $K = \varnothing$ in the formulas of Theorem (2.2.19). □

Remark 2.2.22 It is well-known that the map $x \mapsto xw_{\circ}$ is an orthocomplementation of the weak order: it is involutive, order-reversing and satisfies $xw_{\circ} \wedge x = e$

and $xw_\circ \vee x = w_\circ$. In other words, it endows the weak order with a structure of ortholattice, see for instance (Björner & Brenti, 2005, Corollary 3.2.2). This is no longer the case for the facial weak order: the map $xW_I \longmapsto w_\circ x w_{\circ,I} W_I$ is indeed involutive and order-reversing, but is not an orthocomplementation: for a (counter-)example, consider $x = e$ and $I = S$.

2.2.5 Further properties of the facial weak order

In this section, we study some properties of the facial weak order: we compute its partial Möbius function, discuss formulas for the root inversion sets of meet and join, and describe its join-irreducible elements.

Möbius function

Recall that the *Möbius function* of a poset P is the function $\mu : P \times P \to \mathbb{Z}$ defined inductively by

$$\mu(p,q) := \begin{cases} 1 & \text{if } p = q, \\ -\sum_{p \le r < q} \mu(p,r) & \text{if } p < q, \\ 0 & \text{otherwise.} \end{cases}$$

We refer the reader to (Stanley, 2011) for more information on Möbius functions. The following statement gives the values $\mu(yW_J) := \mu(eW_\emptyset, yW_J)$ of the Möbius function on the facial weak order.

Proposition 2.2.23 *The Möbius function of the facial weak order is given by*

$$\mu(yW_J) := \mu(eW_\varnothing, yW_J) = \begin{cases} (-1)^{|J|}, & \text{if } y = e, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We first show the equality when $y = e$. From the characterization of Theorem (2.2.14) (iii), we know that $xW_I \leq W_J$ if and only if $x = e$ and $I \subseteq J$. That is, the facial weak order below W_J is isomorphic to the boolean lattice on J (for example, the reader can observe a 3-dimensional cube below W in Figure (2.5)). The result follows when $y = e$ since the Möbius function of the boolean lattice on J is given by $\mu(I) = (-1)^{|I|}$ for $I \subseteq J$ (inclusion-exclusion principle (Stanley, 2011)).

We now prove by double induction on the length $\ell(y)$ and the rank $|J|$ that $\mu(yW_J) = 0$ for any coset yW_J with $\ell(y) \geq 1$ and $J \subseteq S$. Indeed, consider yW_J with $\ell(y) \geq 1$ and assume that we have proved that $\mu(xW_I) = 0$ for all xW_I with $1 \leq \ell(x) < \ell(y)$ or with $\ell(x) = \ell(y)$ and $|I| < |J|$. Since $xW_I < yW_J$ implies that $x < y$, or $x = y$ and $I \subsetneq J$ we have

$$\mu(yW_J) = -\sum_{xW_I < yW_J} \mu(xW_I) = -\sum_{W_I < yW_J} \mu(W_I).$$

Therefore, since $W_I < yW_J \iff w_{\circ,I} \leq yw_{\circ,J} \iff I \subseteq D_L(yw_{\circ,J})$, we have the non-empty boolean lattice on $D_L(yw_{\circ,J})$ and

$$\mu(yW_J) = -\sum_{W_I < yW_J} \mu(W_I) = -\sum_{I \subseteq D_L(yw_{\circ,J})} \mu(W_I) = -\sum_{I \subseteq D_L(yw_{\circ,J})} (-1)^{|I|} = 0.$$

□

Formulas for root inversion sets of meet and join

For $X \subseteq \Phi$, define the operators

$$\left[X\right]^{\oplus} := \Phi^+ \cap \operatorname{cone}(X) \qquad \text{and} \qquad \left[X\right]^{\ominus} := \Phi^- \cap \operatorname{cone}(X),$$

and their counterparts

$$\left[X\right]_{\oplus} := \Phi^+ \smallsetminus \left[\Phi^+ \smallsetminus X\right]^{\oplus} \qquad \text{and} \qquad \left[X\right]_{\ominus} := \Phi^- \smallsetminus \left[\Phi^- \smallsetminus X\right]^{\ominus}.$$

Note that $\left[X \cap \Phi^+\right]^{\oplus} = \left[X\right]^{\oplus}$, $\left[X \cap \Phi^+\right]_{\oplus} = \left[X\right]_{\oplus}$, and $-\left[X\right]^{\oplus} = \left[-X\right]^{\ominus}$. Similar formulas hold exchanging $\ominus$'s with $\oplus$'s. Using these notations it is well known that the inversion sets of the meet and join in the (classical) weak order can be computed by

$$\mathbf{N}(x \wedge y) = \left[\mathbf{N}(x) \cap \mathbf{N}(y)\right]_{\oplus} \qquad \text{and} \qquad \mathbf{N}(x \vee y) = \left[\mathbf{N}(x) \cup \mathbf{N}(y)\right]^{\oplus}. \qquad (\heartsuit)$$

For references on this property, see for example (Björner et al., 1990, Theorem 5.5) and the discussion in (Hohlweg & Labbé, 2016) for its extension to infinite Coxeter groups. Our next statement extends these formulas to compute the root inversion sets of the meet and join in the classical weak order.

Corollary 2.2.24 *For $x, y \in W$, the root inversion sets of the meet and join of x and y are given by*

$$\begin{aligned} &\mathbf{R}(x \wedge y) = \left[\mathbf{R}(x) \cup \mathbf{R}(y)\right]^{\ominus} \cup \left[\mathbf{R}(x) \cap \mathbf{R}(y)\right]_{\oplus}, \\ \textit{and} \quad &\mathbf{R}(x \vee y) = \left[\mathbf{R}(x) \cap \mathbf{R}(y)\right]_{\ominus} \cup \left[\mathbf{R}(x) \cup \mathbf{R}(y)\right]^{\oplus}. \end{aligned}$$

Proof. This is immediate from Equation ($\heartsuit$) and Proposition (2.2.10) (i). □

We would now like to compute the root inversion sets of the meet $xW_I \wedge yW_J$ and join $xW_I \vee yW_J$ in the facial weak order in terms of the root inversion sets of xW_I and yW_J. However, we only have a partial answer to this question.

Proposition 2.2.25 *For any cosets $xW_I, yW_J \in \mathcal{P}_W$, we have*

$$\begin{aligned} &\mathbf{R}(xW_I \wedge yW_J) \cap \Phi^- = \left[\mathbf{R}(xW_I) \cup \mathbf{R}(yW_J)\right]^{\ominus}, \\ \textit{and} \quad &\mathbf{R}(xW_I \wedge yW_J) \cap \Phi^+ \subseteq \left[\mathbf{R}(xw_I) \cap \mathbf{R}(yW_J)\right]_{\oplus}, \end{aligned}$$

while

$$\begin{aligned} &\mathbf{R}(xW_I \vee yW_J) \cap \Phi^- \subseteq \left[\mathbf{R}(xW_I) \cap \mathbf{R}(yW_J)\right]_{\ominus}, \\ \textit{and} \quad &\mathbf{R}(xW_I \vee yW_J) \cap \Phi^+ = \left[\mathbf{R}(xW_I) \cup \mathbf{R}(yW_J)\right]^{\oplus}. \end{aligned}$$

Proof. According to Theorem (2.2.19), we have $xW_I \wedge yW_J = z_\wedge W_{K_\wedge}$ with $z_\wedge = x \wedge y$ and $K_\wedge = D_L\Big(z_\wedge^{-1}(xw_{\circ,I} \wedge yw_{\circ,J})\Big)$. Then using Corollaries (2.2.24) and (2.2.13), we have

$$\begin{aligned}\mathbf{R}(xW_I \wedge yW_J) \cap \Phi^- = \mathbf{R}(z_\wedge W_{K_\wedge}) \cap \Phi^- = \mathbf{R}(z_\wedge) \cap \Phi^- = \mathbf{R}(x \wedge y) \cap \Phi^- \\ = \Big[\mathbf{R}(x) \cup \mathbf{R}(y)\Big]^\ominus = \Big[\mathbf{R}(xW_I) \cup \mathbf{R}(yW_J)\Big]^\ominus.\end{aligned}$$

Moreover, since $z_\wedge w_{\circ,K_\wedge} \leq xw_{\circ,I} \wedge yw_{\circ,J}$, we have

$$\begin{aligned}\mathbf{R}(xW_I \wedge yW_J) \cap \Phi^+ = \mathbf{R}(z_\wedge W_{K_\wedge}) \cap \Phi^+ = \mathbf{R}(z_\wedge w_{\circ,K_\wedge}) \cap \Phi^+ \subseteq \mathbf{R}(xw_{\circ,I} \wedge yw_{\circ,J}) \cap \Phi^+ \\ = \Big[\mathbf{R}(xw_{\circ,I}) \cap \mathbf{R}(yw_{\circ,J})\Big]_\oplus = \Big[\mathbf{R}(xW_I) \cap \mathbf{R}(yW_J)\Big]_\oplus\end{aligned}$$

The proof is similar for the join, or can be obtained by the anti-automorphism of Proposition (2.2.18). □

Remark 2.2.26 The inclusions $\mathbf{R}(xW_I \wedge yW_J) \cap \Phi^+ \subseteq \Big[\mathbf{R}(xw_I) \cap \mathbf{R}(yW_J)\Big]_\oplus$ and $\mathbf{R}(xW_I \vee yW_J) \cap \Phi^- \subseteq \Big[\mathbf{R}(xW_I) \cap \mathbf{R}(yW_J)\Big]_\ominus$ can be strict. For example, letting $\alpha = \alpha_r$, $\beta = \alpha_s$, and $\gamma = \alpha_t$, we have by Example (2.2.20)

$$\mathbf{R}(tsrW_{st} \wedge rtsW_\varnothing) \cap \Phi^+ = \mathbf{R}(tW_r) \cap \Phi^+ = \{\alpha, \gamma\}$$

which differs from

$$\Big[\mathbf{R}(tsrW_{st}) \cap \mathbf{R}(rtsW_\varnothing)\Big]_\oplus = \Big[\{\alpha, \gamma, \alpha+\beta+\gamma, -\beta, -\alpha-\beta\}\Big]_\oplus = \{\alpha, \gamma, \alpha+\beta+\gamma\}.$$

Following (Krob et al., 2001, Proposition 9), the set $\mathbf{R}(xW_I \wedge yW_J) \cap \Phi^+$ can be computed as

$$\mathbf{R}(xW_I \wedge yW_J) \cap \Phi^+ = \bigcap_R R \cap \Phi^+$$

where the intersection runs over all $R \subseteq \Phi$ satisfying the equivalent conditions of Corollary (2.2.9) such that

$$R \cap \Phi^+ \subseteq \Big[\mathbf{R}(xw_I) \cap \mathbf{R}(yW_J)\Big]_\oplus \quad \text{while} \quad R \cap \Phi^- \supseteq \Big[\mathbf{R}(xW_I) \cup \mathbf{R}(yW_J)\Big]^\ominus.$$

However, this formula does not provide an efficient way to compute the root set of meets and joins in the facial weak order.

Join irreducible elements

An element x of a finite lattice L is *join-irreducible* if it cannot be written as $x = \bigvee Y$ for some $Y \subseteq L \smallsetminus \{x\}$. Equivalently, x is join-irreducible if it covers exactly one element $x_\star$ of L. For example, the join-irreducible elements of the classical weak order are the elements of W with a single descent. Meet-irreducible elements are defined similarly. We now characterize the join-irreducible elements of the facial weak order.

Proposition 2.2.27 *A coset xW_I is join-irreducible in the facial weak order if and only if $I = \varnothing$ and x is join-irreducible, or $I = \{s\}$ and xs is join-irreducible.*

Proof. Since xW_I covers $xW_{I \smallsetminus \{s\}}$ for any $s \in S$, we have $|I| \leq 1$ for any join-irreducible coset xW_I.

Suppose $I = \varnothing$. The cosets covered by $xW_\varnothing$ are precisely the cosets $xsW_{\{s\}}$ with $xs < x$. Therefore, $xW_\varnothing$ is join-irreducible if and only if x is join-irreducible. Moreover, $(xW_\varnothing)_\star = \{x_\star, x\} = xsW_s$.

Suppose $I = \{s\}$. The cosets covered by $xW_{\{s\}}$ are precisely $xW_\varnothing$ and the cosets $xsw_{\circ,\{s,t\}}W_{\{s,t\}}$ for $xst < xs$. Therefore, $xW_{\{s\}}$ is join-irreducible if and only if xs only covers x, *i.e.*, if xs is join-irreducible. Moreover, $(xW_{\{s\}})_\star = \{x\}$. $\square$

Using the anti-automorphism of Proposition (2.2.18), we get the following statement.

Corollary 2.2.28 *A coset xW_I is meet-irreducible in the facial weak order if and only if $I = \varnothing$ and x is meet-irreducible, or $I = \{s\}$ and x is meet-irreducible.*

2.2.6 Facial weak order on the Davis complex for infinite Coxeter groups

A natural question is to wonder if a facial weak order exists for the Coxeter complex of an infinite Coxeter system (W, S). As the definition given by P. Palacios and M. Ronco makes extensive use of the longest element of each W_I and of the longest element in the coset W^I, we do not know of any way to extend our results to the Coxeter complex.

However, partial results could be obtained with the *Davis complex*, see for instance (Davis, 2008) or (Abramenko & Brown, 2008), which is the simplicial complex

$$\mathcal{D}_W = \bigcup_{\substack{I \subseteq S \\ W_I \text{ finite}}} W/W_I.$$

Definition 2.2.29 Call *facial weak order* $(\mathcal{D}_W, \leq)$ on the Davis complex the order defined by $xW_I \leq yW_J$ if and only if $x \leq y$ and $xw_\circ I \leq yw_\circ J$ in right weak order.

All the results used to prove the existence and formula for the meet in Theorem (2.2.19) in the case of a finite Coxeter system only use the above definition of the facial weak order, as well as standard results valid for any Coxeter system; the finiteness of $W_{K_\wedge}$ being guaranteed by the fact that a standard parabolic subgroup W_K is finite if and only if there is $w \in W$ such that $D_L(w) = K$, see for instance (Björner & Brenti, 2005, Proposition 2.3.1). We therefore obtain the following result that generalizes A. Björner's result for the weak order (Björner, 1984) to the facial weak order on the Davis complex.

Theorem 2.2.30 *The facial weak order on the Davis complex is a meet-semilattice. The meet of two cosets xW_I and yW_J in $\mathcal{D}_W$ is*

$$xW_I \wedge yW_J = z_\wedge W_{K_\wedge} \quad \textit{where} \quad z_\wedge = x \wedge y \quad \textit{and} \quad K_\wedge = D_L\Big(z_\wedge^{-1}(xw_{\circ,I} \wedge yw_{\circ,J})\Big).$$

2.3 Lattice congruences and quotients of the facial weak order

A *lattice congruence* on a lattice $(L, \leq, \wedge, \vee)$ is an equivalence relation $\equiv$ which respects meets and joins, meaning that $x \equiv x'$ and $y \equiv y'$ implies $x \wedge y \equiv x' \wedge y'$ and $x \vee y \equiv x' \vee y'$. Note that a lattice congruence $\equiv$ on L yields a lattice congruence $\boldsymbol{\equiv}$ on the lattice of intervals $\mathcal{I}(L)$, defined by $[x, X] \boldsymbol{\equiv} [x', X'] \iff x \equiv x'$ and $X \equiv X'$.

In this section we start from any lattice congruence $\equiv$ of the weak order and consider the equivalence relation $\boldsymbol{\equiv}$ on the Coxeter complex $\mathcal{P}_W$ by $xW_I \boldsymbol{\equiv} yW_J \iff x \equiv y$ and $xw_{\circ,I} \equiv yw_{\circ,J}$. The goal of this section is to show that $\boldsymbol{\equiv}$ always defines a lattice congruence of the facial weak order. This will require some technical results on the weak order congruence $\equiv$ (see Section (2.3.2)) and on the projection maps of the congruence $\boldsymbol{\equiv}$ (see Theorem (2.3.11)).

On the geometric side, the congruence $\boldsymbol{\equiv}$ of the facial weak order provides a complete description (see Theorem (2.3.22)) of the simplicial fan $\mathcal{F}_\equiv$ associated to the weak order congruence $\equiv$ in N. Reading's work (Reading, 2005): while the classes of $\equiv$ correspond to maximal cones in $\mathcal{F}_\equiv$, the classes of $\boldsymbol{\equiv}$ correspond to all cones in $\mathcal{F}_\equiv$ (maximal or not). We illustrate this construction in Section (2.3.7) with the facial boolean lattice (faces of a cube) and with the facial Cambrian lattices (faces of generalized associahedra) arising from the Cambrian lattices and fans of (Reading, 2006; Reading & Speyer, 2009).

2.3.1 Lattice congruences and projection maps

We first recall the definition of lattice congruences and quotients and refer to (Reading, 2004; Reading, 2006) for further details.

Definition 2.3.1 An *order congruence* is an equivalence relation $\equiv$ on a poset P

such that:

(i) Every equivalence class under $\equiv$ is an interval of P.

(ii) The projection $\pi^{\uparrow} : P \to P$ (resp. $\pi_{\downarrow} : P \to P$), which maps an element of P to the maximal (resp. minimal) element of its equivalence class, is order preserving.

The *quotient* $P/{\equiv}$ is a poset on the equivalence classes of $\equiv$, where the order relation is defined by $X \leq Y$ in $P/{\equiv}$ if and only if there exist representatives $x \in X$ and $y \in Y$ such that $x \leq y$ in P. The quotient $P/{\equiv}$ is isomorphic to the subposet of P induced by $\pi_{\downarrow}(P)$ (or equivalently by $\pi^{\uparrow}(P)$).

If, moreover, P is a finite lattice, then $\equiv$ is a lattice congruence, meaning that it is compatible with meets and joins: for any $x \equiv x'$ and $y \equiv y'$, we have $x \wedge y \equiv x' \wedge y'$ and $x \vee y \equiv x' \vee y'$. The poset quotient $P/{\equiv}$ then inherits a lattice structure where the meet $X \wedge Y$ (resp. the join $X \vee Y$) of two congruence classes X and Y is the congruence class of $x \wedge y$ (resp. of $x \vee y$) for arbitrary representatives $x \in X$ and $y \in Y$.

In our constructions we will use the projection maps $\pi^{\uparrow}$ and $\pi_{\downarrow}$ to define congruences. By definition note that $\pi_{\downarrow}(x) \leq x \leq \pi^{\uparrow}(x)$, that $\pi^{\uparrow} \circ \pi^{\uparrow} = \pi^{\uparrow} \circ \pi_{\downarrow} = \pi^{\uparrow}$ while $\pi_{\downarrow} \circ \pi_{\downarrow} = \pi_{\downarrow} \circ \pi^{\uparrow} = \pi_{\downarrow}$, and that $\pi^{\uparrow}$ and $\pi_{\downarrow}$ are order preserving. The following lemma shows the reciprocal statement.

Lemma 2.3.2 *If two maps $\pi^{\uparrow} : P \to P$ and $\pi_{\downarrow} : P \to P$ satisfy*

(i) $\pi_{\downarrow}(x) \leq x \leq \pi^{\uparrow}(x)$ *for any element* $x \in P$,

(ii) $\pi^{\uparrow} \circ \pi^{\uparrow} = \pi^{\uparrow} \circ \pi_{\downarrow} = \pi^{\uparrow}$ *and* $\pi_{\downarrow} \circ \pi_{\downarrow} = \pi_{\downarrow} \circ \pi^{\uparrow} = \pi_{\downarrow}$,

(iii) $\pi^{\uparrow}$ *and* $\pi_{\downarrow}$ *are order preserving,*

then the fibers of $\pi^{\uparrow}$ *and* $\pi_{\downarrow}$ *coincide and the relation* $\equiv$ *on* P *defined by*

$$x \equiv y \iff \pi^{\uparrow}(x) = \pi^{\uparrow}(y) \iff \pi_{\downarrow}(x) = \pi_{\downarrow}(y)$$

is an order congruence on P *with projection maps* $\pi^{\uparrow}$ *and* $\pi_{\downarrow}$*.*

Proof. First, Condition (ii) ensures that $\pi^{\uparrow}(x) = \pi^{\uparrow}(y) \iff \pi_{\downarrow}(x) = \pi_{\downarrow}(y)$ for any $x, y \in P$, so that the fibers of the maps $\pi^{\uparrow}$ and $\pi_{\downarrow}$ coincide. We now claim that if $z \in \pi_{\downarrow}(P)$, then the fiber $\pi_{\downarrow}^{-1}(z)$ is the interval $[z, \pi^{\uparrow}(z)]$. Indeed, if $\pi_{\downarrow}(x) = z$, then $\pi^{\uparrow}(x) = \pi^{\uparrow}(\pi_{\downarrow}(x)) = \pi^{\uparrow}(z)$ by Condition (ii), so that $z \leq x \leq \pi^{\uparrow}(z)$ by Condition (i). Reciprocally, for any $z \leq x \leq \pi^{\uparrow}(z)$, Conditions (ii) and (iii) ensure that $z = \pi_{\downarrow}(z) \leq \pi_{\downarrow}(x) \leq \pi_{\downarrow}(\pi^{\uparrow}(z)) = \pi_{\downarrow}(z) = z$, so that $\pi_{\downarrow}(x) = z$. We conclude that the fibers of $\pi^{\uparrow}$ (or equivalently of $\pi_{\downarrow}$) are intervals of P, and that $\pi^{\uparrow}$ (resp. $\pi_{\downarrow}$) indeed maps an element of P to the maximal (resp. minimal) element of its fiber. Since $\pi^{\uparrow}$ and $\pi_{\downarrow}$ are order preserving, this shows that the fibers indeed define an order congruence. □

2.3.2 Congruences of the weak order

Consider a lattice congruence $\equiv$ of the weak order whose up and down projections are denoted by $\pi^{\uparrow}$ and $\pi_{\downarrow}$ respectively. We will need the following elementary properties of $\equiv$. Recall, the notation xW_I means that we are considering x in W^I.

Lemma 2.3.3 *For any coset* xW_I *and any* $s \in I$*, we have* $x \equiv xs \iff xsw_{\circ,I} \equiv xw_{\circ,I}$*.*

Proof. Assume $x \equiv xs$. As $x \in W^I$ and $s \in I$, we have $xs \not\leq xsw_{\circ,I}$. Therefore, $xsw_{\circ,I} = x \vee xsw_{\circ,I} \equiv xs \vee xsw_{\circ,I} = xw_{\circ,I}$. The reverse implication can be proved similarly or applying the anti-automorphism $x \to xw_{\circ}$. □

We will need a refined version of the previous lemma for cosets of a rank 2 parabolic subgroup. Consider a coset $xW_{\{s,t\}}$ with $s,t \in S \setminus D_R(x)$. It consist of two chains

$$x \leq xs \leq \cdots \leq xtw_{\circ,\{s,t\}} \leq xw_{\circ,\{s,t\}} \quad \text{and} \quad x \leq xt \leq \cdots \leq xsw_{\circ,\{s,t\}} \leq xw_{\circ,\{s,t\}}$$

from x to $xw_{\circ,\{s,t\}}$. The following two lemmas are of same nature: they state that a single congruence between two elements of $xW_{\{s,t\}}$ can force almost all elements in $xW_{\{s,t\}}$ to be congruent. These lemmas are illustrated in Figure (2.8).

Lemma 2.3.4 *For any coset $xW_{\{s,t\}}$, if $x \equiv xs$ or $xsw_{\circ,\{s,t\}} \equiv xw_{\circ,\{s,t\}}$ then*

$$x \equiv xs \equiv xst \equiv \cdots \equiv xtw_{\circ,\{s,t\}} \quad \textit{and} \quad xt \equiv xts \equiv \cdots \equiv xsw_{\circ,\{s,t\}} \equiv xw_{\circ,\{s,t\}}.$$

Proof. Assume $x \equiv xs$. For any $xt \leq y \leq xw_{\circ,\{s,t\}}$ we have $xs \vee y = xw_{\circ,\{s,t\}}$. Since $x \equiv xs$ and $\equiv$ is a lattice congruence, we get $y = x \vee y \equiv xs \vee y = xw_{\circ,\{s,t\}}$. Now for any $x \leq z \leq xtw_{\circ,\{s,t\}}$, we have $y \wedge z = x$. Since $y \equiv xw_{\circ,\{s,t\}}$ and $\equiv$ is a lattice congruence, we get $z = xw_{\circ,\{s,t\}} \wedge z \equiv y \wedge z = x$. The proof is similar if we assume instead $xsw_{\circ,\{s,t\}} \equiv xw_{\circ,\{s,t\}}$. ☐

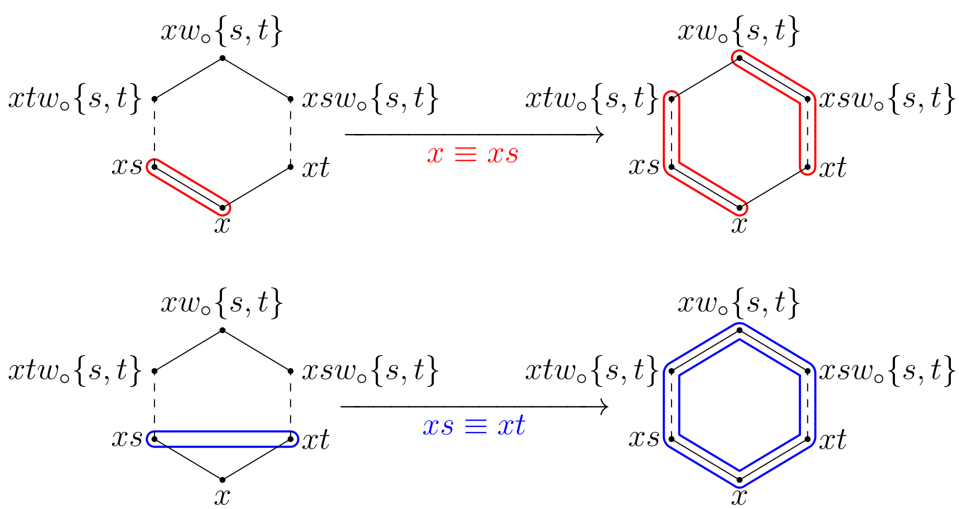


Figure 2.8: In a coset $xW_{\{s,t\}}$, a single congruence may force many congruences. See Lemma (2.3.4) for the congruence $x \equiv xs$ (top) and Lemma (2.3.5) for the congruence $xs \equiv xt$ (bottom).

Lemma 2.3.5 *For any coset $xW_{\{s,t\}}$, if $xs \equiv xt$ then $x \equiv y$ for all $y \in xW_{\{s,t\}}$.*

Proof. Since $xs \equiv xt$, we have $\pi_\downarrow(xs) = \pi_\downarrow(xt)$ and $\pi^\uparrow(xs) = \pi^\uparrow(xt)$. Using that $\pi_\downarrow(xs) \leq xs \leq \pi^\uparrow(xs)$ and $\pi_\downarrow(xt) \leq xt \leq \pi^\uparrow(xs)$, we obtain that

$$\pi_\downarrow(xs) \leq xs \wedge xt = x \leq xw_{\circ,\{s,t\}} = xs \vee xt \leq \pi^\uparrow(xs).$$

Since the congruence class of xs is the interval $[\pi_\downarrow(xs), \pi^\uparrow(xs)]$, it certainly contains all the coset $xW_{\{s,t\}}$. We conclude that $x \equiv y$ for all $y \in xW_{\{s,t\}}$. □

Throughout the end of this section, we write $x \leqq y$ when $x \leq y$ and $x \equiv y$. In other words, $x \leqq y \iff x \leq y \leq \pi^\uparrow(x) \iff \pi_\downarrow(y) \leq x \leq y$. Note that the relation $\leqq$ is transitive (as the intersection of two transitive relations) and stable by meet and join (as $\leq$ is a lattice and $\equiv$ a lattice congruence).

The goal of the following statements is to show that one can "translate faces along congruence classes". We make this statement precise in the following lemmas. The next lemma is a rephrasing of (Reading, 2005, Proposition 2.2) with an alternative proof.

Lemma 2.3.6 *For any $x \in W$ and $t \in S \smallsetminus D_R(x)$ such that $x \not\equiv xt$, there exists a unique $\sigma^\uparrow(x,t) \in S \smallsetminus D_R\big(\pi^\uparrow(x)\big)$ such that $xt \leqq \pi^\uparrow(x)\sigma^\uparrow(x,t)$.*

Proof. To prove the existence of $\sigma^\uparrow(x,t)$, we work by induction on the length of a minimal path from x to $\pi^\uparrow(x)$ in weak order. If $x = \pi^\uparrow(x)$, then $\sigma^\uparrow(x,t) = t$ meets our criteria. We now assume that there exists $s \in S \smallsetminus D_R(x)$ such that $x \leqq xs \leqq \pi^\uparrow(x)$. Let $x' = xtw_{\circ,\{s,t\}}$ and $t' = w_{\circ,\{s,t\}}tw_{\circ,\{s,t\}}$. We get from Lemma (2.3.4) that $x \equiv x'$, thus $\pi^\uparrow(x) = \pi^\uparrow(x')$ and $xt \leqq xw_{\circ,\{s,t\}} = x't'$. Since $x \not\equiv xt$, this also ensures that $x' \not\equiv x't'$. Thus the length of a minimal path from between $x't'$ and $\pi^\uparrow(x')$ is strictly smaller than the length of a minimal path between x and $\pi^\uparrow(x)$. Therefore, by induction hypothesis, there exists $\sigma^\uparrow(x',t') \in S$ such that $x't' \leqq \pi^\uparrow(x')\sigma^\uparrow(x',t')$.

We therefore obtain that

$$xt \leqq x't' \leqq \pi^{\uparrow}(x')\sigma^{\uparrow}(x',t') = \pi^{\uparrow}(x)\sigma^{\uparrow}(x',t'),$$

and conclude that $\sigma^{\uparrow}(x,t) = \sigma^{\uparrow}(x',t')$ meets our criteria.

To prove uniqueness, assume that there exist $r \neq s \in S \smallsetminus D_R(y)$ which both satisfy $xt \leqq \pi^{\uparrow}(x)r$ and $xt \leqq \pi^{\uparrow}(x)s$. This implies that $\pi^{\uparrow}(x)r \equiv xt \equiv \pi^{\uparrow}(x)s$, so that $\pi^{\uparrow}(x) \equiv \pi^{\uparrow}(x)r \equiv \pi^{\uparrow}(x)s$ by application of Lemma (2.3.5). We would therefore obtain that $x \equiv \pi^{\uparrow}(x) \equiv \pi^{\uparrow}(x)r \equiv xt$, a contradiction. □

Lemma 2.3.7 *For any coset xW_I, the set $\Sigma^{\uparrow}(x,I) := \left\{\sigma^{\uparrow}(x,t) \mid t \in I,\ x \not\equiv xt\right\}$ is the unique subset of $S \smallsetminus D_R\big(\pi^{\uparrow}(x)\big)$ such that $xw_{\circ,I} \leqq \pi^{\uparrow}(x)w_{\circ,\Sigma^{\uparrow}(x,I)}$.*

Proof. Split I into $I_{\equiv} \sqcup I_{\not\equiv}$ where $I_{\equiv} := \{t \in I \mid x \equiv xt\}$ and $I_{\not\equiv} := \{t \in I \mid x \not\equiv xt\}$. Since $\leqq$ is stable by join, we get

$$xw_{\circ,I} = \left(\bigvee_{t \in I_{\equiv}} xt\right) \vee \left(\bigvee_{t \in I_{\not\equiv}} xt\right) \leqq \pi^{\uparrow}(x) \vee \left(\bigvee_{t \in I_{\not\equiv}} \pi^{\uparrow}(x)\sigma^{\uparrow}(x,t)\right) = \pi^{\uparrow}(x)w_{\circ,\Sigma^{\uparrow}(x,I)}.$$

To prove unicity, we observe that there already is a unique maximal subset Σ of $S \smallsetminus D_R\big(\pi^{\uparrow}(x)\big)$ such that $xw_{\circ,I} \leqq \pi^{\uparrow}(x)w_{\circ,\Sigma}$ since $\leqq$ is stable by join. Consider now any subset Σ' of Σ with this property. Since $\pi^{\uparrow}(x)w_{\circ,\Sigma'} \equiv xw_{\circ,I} \equiv \pi^{\uparrow}(x)w_{\circ,\Sigma}$, we obtain

$$\pi^{\uparrow}(x)w_{\circ,\Sigma\smallsetminus\Sigma'} = \pi^{\uparrow}(x)w_{\circ,\Sigma\smallsetminus\Sigma'} \wedge \pi^{\uparrow}(x)w_{\circ,\Sigma} \equiv \pi^{\uparrow}(x)w_{\circ,\Sigma\smallsetminus\Sigma'} \wedge \pi^{\uparrow}(x)w_{\circ,\Sigma'} = \pi^{\uparrow}(x).$$

Since $\pi^{\uparrow}(x)$ is maximal in its congruence class, this implies that $w_{\circ,\Sigma\smallsetminus\Sigma'} = e$ so that $\Sigma' = \Sigma$. □

Using similar arguments as in the previous lemmas, or applying the anti-automorphism $x \mapsto xw_{\circ}$, we deduce the following statement, similar to Lemma (2.3.7).

Lemma 2.3.8 *For any coset xW_I, there is a unique subset $\Sigma_{\downarrow}(x,I)$ of $D_R\big(\pi_{\downarrow}(xw_{\circ,I})\big)$ such that $\pi_{\downarrow}(xw_{\circ,I})w_{\circ,\Sigma_{\downarrow}(x,I)} \leqq x$.*

Remark 2.3.9 Consider a coset xW_I. If $\pi^{\uparrow}(x) = x$ then $\sigma^{\uparrow}(x,t) = t$ for all $t \in I$ and thus $\Sigma^{\uparrow}(x,I) = I$. Similarly, if $\pi_{\downarrow}(xw_{\circ,I}) = xw_{\circ,I}$ then $\Sigma_{\downarrow}(x,I) = I$.

We will furthermore need the following properties of $\sigma^{\uparrow}(x,t)$ and $\Sigma^{\uparrow}(x,I)$.

Lemma 2.3.10 *For any coset xW_I and any $t \in I$, we have*

- *if $x \equiv xt$ then $xtw_{\circ,I} \equiv \pi^{\uparrow}(x)w_{\circ,\Sigma^{\uparrow}(x,I)}$,*
- *if $x \not\equiv xt$ then $xtw_{\circ,I} \equiv \pi^{\uparrow}(x)\sigma^{\uparrow}(x,t)w_{\circ,\Sigma^{\uparrow}(x,I)}$.*

In other words, either

$$xtw_{\circ,I} \equiv xw_{\circ,I} \equiv \pi^{\uparrow}(x)w_{\circ,\Sigma^{\uparrow}(x,I)} \quad \text{or} \quad xtw_{\circ,I} \equiv \pi^{\uparrow}(x)\sigma^{\uparrow}(x,t)w_{\circ,\Sigma^{\uparrow}(x,I)}.$$

Proof. If $x \equiv xt$, Lemmas (2.3.3) and (2.3.7) ensure that $xtw_{\circ,I} \equiv xw_{\circ,I} \equiv \pi^{\uparrow}(x)w_{\circ,\Sigma^{\uparrow}(x,I)}$. Assume now that $x \not\equiv xt$. Observe that we have $\pi_{\downarrow}(\pi^{\uparrow}(x)w_{\circ,\Sigma^{\uparrow}(x,I)}) = \pi_{\downarrow}(xw_{\circ,I})$ since $xw_{\circ,I} \equiv \pi^{\uparrow}(x)w_{\circ,\Sigma^{\uparrow}(x,I)}$. Consider the subsets

$$X := \Sigma_{\downarrow}(x,I) \qquad \text{and} \qquad Y := \Sigma_{\downarrow}(\pi^{\uparrow}(x), \Sigma^{\uparrow}(x,I)).$$

of $D_R(\pi_{\downarrow}(xw_{\circ,I}))$. By definition of $\Sigma_{\downarrow}$, we have

$$\pi_{\downarrow}(xw_{\circ,I})w_{\circ,X} \equiv x \equiv \pi^{\uparrow}(x) \equiv \pi_{\downarrow}(\pi^{\uparrow}(x)w_{\circ,\Sigma^{\uparrow}(x,I)})w_{\circ,Y} = \pi_{\downarrow}(xw_{\circ,I})w_{\circ,Y},$$

We therefore obtain that

$$\pi_{\downarrow}(xw_{\circ,I})w_{\circ,X\cup Y} = \pi_{\downarrow}(xw_{\circ,I})w_{\circ,X} \wedge \pi_{\downarrow}(xw_{\circ,I})w_{\circ,Y} \equiv \pi_{\downarrow}(xw_{\circ,I})w_{\circ,X}$$

which in turns implies that $Y \subseteq X$. It follows that there is t' such that

$$xt'w_{\circ,I} \equiv \pi^{\uparrow}(x)\sigma^{\uparrow}(x,t)w_{\circ,\Sigma^{\uparrow}(x,I)}.$$

Observe that

$$xt'w_{\circ,I} \equiv \pi^{\uparrow}(x)\sigma^{\uparrow}(x,t)w_{\circ,\Sigma^{\uparrow}(x,I)} = \pi^{\uparrow}(x)\sigma^{\uparrow}(x,t)w_{\circ,\Sigma^{\uparrow}(x,I)} \vee \pi^{\uparrow}(x)\sigma^{\uparrow}(x,s) \equiv xt'w_{\circ,I} \vee xs$$

for all $s \in I \smallsetminus \{t\}$ such that $x \not\equiv xs$. Since $xt'w_{\circ,I} \not\equiv xw_{\circ,I}$ by Lemma (2.3.3), we obtain that $t' = t$ and therefore $xtw_{\circ,I} \equiv \pi^{\uparrow}(x)\sigma^{\uparrow}(x,t)w_{\circ,\Sigma^{\uparrow}(x,I)}$. $\square$

2.3.3 Congruences of the facial weak order

Based on the properties established in the previous section we now show that the lattice congruences of the weak order naturally extend to lattice congruences of the facial weak order. We start from a lattice congruence $\equiv$ of the weak order whose up and down projections are denoted by $\pi^{\uparrow}$ and $\pi_{\downarrow}$ respectively. We then define two maps $\Pi^{\uparrow} : \mathcal{P}_W \to \mathcal{P}_W$ and $\Pi_{\downarrow} : \mathcal{P}_W \to \mathcal{P}_W$ by

$$\Pi^{\uparrow}(xW_I) = \pi^{\uparrow}(x)W_{\Sigma^{\uparrow}(x,I)} \qquad \text{and} \qquad \Pi_{\downarrow}(xW_I) = \pi_{\downarrow}(xw_{\circ,I})W_{\Sigma_{\downarrow}(x,I)}$$

where $\Sigma^{\uparrow}(x, I)$ and $\Sigma_{\downarrow}(x, I)$ are the subsets of S defined by Lemmas (2.3.7) and (2.3.8). Note that we again take the liberty here to write $\Pi_{\downarrow}(xW_I) = \pi_{\downarrow}(xw_{\circ,I})W_{\Sigma_{\downarrow}(x,I)}$ instead of $\Pi_{\downarrow}(xW_I) = \pi_{\downarrow}(xw_{\circ,I})\, w_{\circ,\Sigma_{\downarrow}(x,I)}W_{\Sigma_{\downarrow}(x,I)}$ to make apparent the symmetry between $\Pi^{\uparrow}$ and $\Pi_{\downarrow}$.

It immediately follows from Lemmas (2.3.7) and (2.3.8) that $\Pi^{\uparrow}(xW_I)$ is the biggest parabolic coset in the interval $[\pi^{\uparrow}(x), \pi^{\uparrow}(xw_{\circ,I})]$ containing $\pi^{\uparrow}(x)$ and similarly $\Pi_{\downarrow}(xW_I)$ is the biggest parabolic coset in the interval $[\pi_{\downarrow}(x), \pi_{\downarrow}(xw_{\circ,I})]$ containing $\pi_{\downarrow}(xw_{\circ,I})$.

Theorem 2.3.11 *The maps $\Pi^{\uparrow}$ and $\Pi_{\downarrow}$ fulfill the following properties:*

(i) $\Pi_{\downarrow}(xW_I) \le xW_I \le \Pi^{\uparrow}(xW_I)$ *for any coset* xW_I.

(ii) $\Pi^{\uparrow} \circ \Pi^{\uparrow} = \Pi^{\uparrow} \circ \Pi_{\downarrow} = \Pi^{\uparrow}$ *and* $\Pi_{\downarrow} \circ \Pi_{\downarrow} = \Pi_{\downarrow} \circ \Pi^{\uparrow} = \Pi_{\downarrow}$.

(iii) $\Pi^{\uparrow}$ *and* $\Pi_{\downarrow}$ *are order preserving.*

Therefore, the fibers of the maps $\Pi^{\uparrow}$ *and* $\Pi_{\downarrow}$ *coincide and define a lattice congruence* $\equiv$ *of the facial weak order.*

Proof. Using the characterization of the facial weak order given in Theorem (2.2.14) (iii), we obtain that $xW_I \le \Pi^{\uparrow}(xW_I)$ since $x \le \pi^{\uparrow}(x)$ and $xw_{\circ,I} \le \pi^{\uparrow}(x)w_{\circ,\Sigma^{\uparrow}(x,I)}$. Similarly, $\Pi_{\downarrow}(xW_I) \le xW_I$ since $\pi_{\downarrow}(xw_{\circ,I})w_{\circ,\Sigma_{\downarrow}(x,I)} \le x$ and $\pi_{\downarrow}(xw_{\circ,I}) \le xw_{\circ,I}$. This shows (i).

For (ii), it follows from the definition that $\Pi^{\uparrow}\Big(\Pi^{\uparrow}(xW_I)\Big) = \Pi^{\uparrow}\Big(\pi^{\uparrow}(x)W_{\Sigma^{\uparrow}(x,I)}\Big)$ is the biggest parabolic coset in the interval $\Big[\pi^{\uparrow}\Big(\pi^{\uparrow}(x)\Big), \pi^{\uparrow}\Big(\pi^{\uparrow}(x)w_{\circ,\Sigma^{\uparrow}(x,I)}\Big)\Big]$ containing $\pi^{\uparrow}\Big(\pi^{\uparrow}(x)\Big)$. However, we have $\pi^{\uparrow}\Big(\pi^{\uparrow}(x)\Big) = \pi^{\uparrow}(x)$ and $\pi^{\uparrow}\Big(\pi^{\uparrow}(x)w_{\circ,\Sigma^{\uparrow}(x,I)}\Big) = \pi^{\uparrow}(xw_{\circ,I})$ since $xw_{\circ,I} \equiv \pi^{\uparrow}(x)w_{\circ,\Sigma^{\uparrow}(x,I)}$. We conclude that $\Pi^{\uparrow} \circ \Pi^{\uparrow} = \Pi^{\uparrow}$. The proof is similar for the other equalities of (ii).

To prove (iii), it is enough to show that $\Pi^{\uparrow}$ is order-preserving on covering relations of the facial weak order (it is then order preserving on any weak order relation by transitivity, and the result for $\Pi_{\downarrow}$ can be argued similarly or using the anti-automorphisms of Proposition (2.2.18)). Therefore, we consider a cover relation $xW_I \lessdot yW_J$ in facial weak order and prove that $\Pi^{\uparrow}(xW_I) \le \Pi^{\uparrow}(yW_J)$.

It is immediate if the cover relation $xW_I \lessdot yW_J$ is of type (1), that is, if $x = y$ and $J = I \cup \{s\}$. Indeed, it follows from the characterization in terms of biggest parabolic subgroups and from the fact that $\pi^{\uparrow}(x) = \pi^{\uparrow}(y)$ and $\pi^{\uparrow}(xw_{\circ,I}) \le \pi^{\uparrow}(yw_{\circ,J})$.

Consider now a cover relation $xW_I \lessdot yW_J$ of type (2), that is, with $y = xw_{\circ,I}w_{\circ,J}$ and $J = I \smallsetminus \{s\}$. Note that in this case $\pi^{\uparrow}(x) \le \pi^{\uparrow}(y)$ and $\pi^{\uparrow}(xw_{\circ,I}) = \pi^{\uparrow}(yw_{\circ,J})$. We therefore need to show that $\pi^{\uparrow}(x)w_{\circ,\Sigma^{\uparrow}(x,I)} \le \pi^{\uparrow}(y)w_{\circ,\Sigma^{\uparrow}(y,J)}$.

For $t \in S$, define $t^{\star} := w_{\circ,I}w_{\circ,J}tw_{\circ,J}w_{\circ,I}$ so that the equality $xw_{\circ,I} = yw_{\circ,J}$ implies the equality $ytw_{\circ,J} = xt^{\star}w_{\circ,I}$. Let

$$J_{\equiv} := \{t \in J \mid \; ytw_{\circ,J} \equiv yw_{\circ,J}\} = \{t \in J \mid \; xt^{\star}w_{\circ,I} \equiv xw_{\circ,I}\},$$

$$J_{\not\equiv} := \{t \in J \mid \; ytw_{\circ,J} \not\equiv yw_{\circ,J}\} = \{t \in J \mid \; xt^{\star}w_{\circ,I} \not\equiv xw_{\circ,I}\},$$

and consider

$$K := \left\{ w_{\circ,\Sigma^{\uparrow}(x,I)} \sigma^{\uparrow}(x, t^{\star}) w_{\circ,\Sigma^{\uparrow}(x,I)} \;\middle|\; t \in J_{\not\equiv} \right\} \qquad \text{and} \qquad z := \pi^{\uparrow}(x) w_{\circ,\Sigma^{\uparrow}(x,I)} w_{\circ,K}.$$

Lemma (2.3.10) ensures that

$$ytw_{\circ,J} = xt^{\star}w_{\circ,I} \equiv \begin{cases} \pi^{\uparrow}(x) w_{\circ,\Sigma^{\uparrow}(x,I)} & \text{if } t \in J_{\equiv}, \\ \pi^{\uparrow}(x) \sigma^{\uparrow}(x, t^{\star}) w_{\circ,\Sigma^{\uparrow}(x,I)} & \text{if } t \in J_{\not\equiv}. \end{cases}$$

Therefore

$$\begin{aligned} y = \bigwedge_{t \in J} ytw_{\circ,J} &= \bigwedge_{t \in J_{\equiv}} ytw_{\circ,J} \wedge \bigwedge_{t \in J_{\not\equiv}} ytw_{\circ,J} \\ &\equiv \pi^{\uparrow}(x) w_{\circ,\Sigma^{\uparrow}(x,I)} \wedge \bigwedge_{t \in J_{\not\equiv}} \pi^{\uparrow}(x) \sigma^{\uparrow}(x, t^{\star}) w_{\circ,\Sigma^{\uparrow}(x,I)} \\ &= \pi^{\uparrow}(x) w_{\circ,\Sigma^{\uparrow}(x,I)} \bigwedge_{t \in J_{\not\equiv}} w_{\circ,\Sigma^{\uparrow}(x,I)} \sigma^{\uparrow}(x, t^{\star}) w_{\circ,\Sigma^{\uparrow}(x,I)} \\ &= \pi^{\uparrow}(x) w_{\circ,\Sigma^{\uparrow}(x,I)} w_{\circ,K} = z. \end{aligned}$$

By Lemma (2.3.7) applied to the coset zW_K, there exists $\Sigma^{\uparrow}(z, K)$ such that

$$\pi^{\uparrow}(x) w_{\circ,\Sigma^{\uparrow}(x,I)} = zw_{\circ,K} \leqq \pi^{\uparrow}(z) w_{\circ,\Sigma^{\uparrow}(z,K)} = \pi^{\uparrow}(y) w_{\circ,\Sigma^{\uparrow}(z,K)}.$$

Since $\pi^{\uparrow}(x) w_{\circ,\Sigma^{\uparrow}(x,I)} \equiv xw_{\circ,I} = yw_{\circ,J}$, it follows that $\Sigma^{\uparrow}(y, J) = \Sigma^{\uparrow}(z, K)$ by uniqueness in Lemma (2.3.7) applied to the coset yW_J. We get that $\pi^{\uparrow}(x) w_{\circ,\Sigma^{\uparrow}(x,I)} \leq \pi^{\uparrow}(y) w_{\circ,\Sigma^{\uparrow}(y,J)}$ and thus that $\Pi^{\uparrow}(xW_I) \leq \Pi^{\uparrow}(yW_J)$.

We conclude by Lemma (2.3.2) that the fibers of $\Pi^{\uparrow}$ and $\Pi_{\downarrow}$ indeed coincide and define a lattice congruence $\equiv$ of the facial weak order. □

2.3.4 Properties of facial congruences

In this section, we gather some properties of the facial congruence $\equiv$ defined in Theorem (2.3.11).

Basic properties

We first come back to the natural definition of $≣$ given in the introduction of Section (2.3).

Proposition 2.3.12 *For any cosets $xW_I, yW_J \in \mathcal{P}_W$,*

$$xW_I ≣ yW_J \iff x \equiv y \text{ and } xw_{\circ,I} \equiv yw_{\circ,J}.$$

Proof. If $xW_I ≣ yW_J$, then $\Pi^{\uparrow}(xW_I) = \Pi^{\uparrow}(yW_J)$ so that $\pi^{\uparrow}(x) = \pi^{\uparrow}(y)$ and $x \equiv y$. Moreover, $\Pi_{\downarrow}(xW_I) = \Pi_{\downarrow}(yW_J)$ so that $\pi_{\downarrow}(xw_{\circ,I}) = \pi_{\downarrow}(yw_{\circ,J})$ and $xw_{\circ,I} \equiv yw_{\circ,J}$. Therefore, the $≣$-congruence class of xW_I determines the $\equiv$-congruence classes of x and of $xw_{\circ,I}$. Reciprocally, we already observed that $\Pi^{\uparrow}(xW_I)$ is the biggest parabolic coset in the interval $[\pi^{\uparrow}(x), \pi^{\uparrow}(xw_{\circ,I})]$ containing $\pi^{\uparrow}(x)$. If $x \equiv y$ and $xw_{\circ,I} \equiv yw_{\circ,J}$, we obtain that $\Pi^{\uparrow}(xW_I) = \Pi^{\uparrow}(yW_J)$. Therefore, the $≣$-congruence class of xW_I only depends on the $\equiv$-congruence classes of x and of $xw_{\circ,I}$. $\square$

Corollary 2.3.13 *For any $x, y \in W$, we have $x \equiv y \iff xW_{\varnothing} ≣ yW_{\varnothing}$. Therefore, each congruence class γ of $\equiv$ is the intersection of W with a congruence class Γ of $≣$.*

This corollary says that the congruence $≣$ of the facial weak order indeed extends the congruence $\equiv$ of the weak order. Nevertheless, observe that not all congruences of the facial weak order arise as congruences of the weak order (consider for instance the congruence on $\mathcal{P}_{A_2}$ that only contracts sW_t with $stW_{\varnothing}$).

Join-irreducible contractions

Recall that an element x of a finite lattice L is *join-irreducible* if it covers exactly one element $x_{\star}$ (see Section (2.2.5)). The following statement can be found *e.g.* in (Freese et al., 1995, Lemma 2.32). For a lattice congruence $\equiv$ on L

and $y \in L$, let $D_{\equiv}(y)$ denote the set of join-irreducible elements $x \leq y$ not contracted by $\equiv$, that is such that $x_\star \not\equiv x$. For y and z in L, we then have that $y \equiv z \iff D_{\equiv}(y) = D_{\equiv}(z)$ and lattice quotient $L/{\equiv}$ is isomorphic to the inclusion poset on $\{D_{\equiv}(y) \mid y \in L\}$. In other words, the lattice congruence $\equiv$ is characterized by the join-irreducible elements of L that it contracts. Even if this characterization is not always convenient, it is relevant to describe the join-irreducibles of the facial weak order contracted by $≣$ in terms of those contracted by $\equiv$.

Proposition 2.3.14 *The join-irreducible cosets of the facial weak order contracted by $≣$ are precisely:*

- *the cosets $xW_\varnothing$ where x is a join-irreducible element of the weak order contracted by $\equiv$,*
- *the cosets $xW_{\{s\}}$ where xs is a join-irreducible element of the weak order contracted by $\equiv$.*

Proof. The join-irreducible cosets of the facial weak order are described in Proposition (2.2.27). Now $xW_\varnothing$ is contracted by $≣$ when $xW_\varnothing ≣ (xW_\varnothing)_\star = \{x_\star, x\}$, that is, when $x \equiv x_\star$ by Proposition (2.3.12). Similarly, $xW_{\{s\}}$ is contracted by $≣$ when $xW_{\{s\}} ≣ (xW_{\{s\}})_\star = \{x\}$, that is, when $xs \equiv x$ by Proposition (2.3.12). □

Up and bottom cosets of facial congruence classes

The next statements deal with maximal and minimal cosets in their facial congruence classes.

Proposition 2.3.15 *For any coset xW_I, we have*

(i) $\Pi^{\uparrow}(xW_I) = xW_I \iff \pi^{\uparrow}(x) = x$,

(ii) $\Pi_{\downarrow}(xW_I) = xW_I \iff \pi_{\downarrow}(xw_{\circ,I}) = xw_{\circ,I}$.

Proof. We only prove (i), the proof of (ii) being symmetric. Recall the definition $\Pi^{\uparrow}(xW_I) = \pi^{\uparrow}(x)W_{\Sigma^{\uparrow}(x,I)}$. Therefore, $\Pi^{\uparrow}(xW_I) = xW_I$ clearly implies that $\pi^{\uparrow}(x) = x$. Reciprocally, if $\pi^{\uparrow}(x) = x$, then $\Sigma^{\uparrow}(x, I) = I$ by the uniqueness of $\Sigma^{\uparrow}(x, I)$ in Lemma (2.3.7). Therefore $\Pi^{\uparrow}(xW_I) = \pi^{\uparrow}(x)W_{\Sigma^{\uparrow}(x,I)} = xW_I$. □

Call an element x in W a $\equiv$-*singleton* if it is alone in its $\equiv$-congruence class, *i.e.*, such that $\pi_{\downarrow}(x) = x = \pi^{\uparrow}(x)$. Similarly, call a coset xW_I a *facial* $\equiv$-*singleton* if it is alone in its $\equiv$-congruence class, *i.e.*, such that $\Pi_{\downarrow}(xW_I) = xW_I = \Pi^{\uparrow}(xW_I)$.

Proposition 2.3.16 *(i) A coset xW_I is a facial $\equiv$-singleton if and only if $\pi^{\uparrow}(x) = x$ and $\pi_{\downarrow}(xw_{\circ,I}) = xw_{\circ,I}$.*

(ii) If x is a $\equiv$-singleton, then xW_I is a facial $\equiv$-singleton for any $I \subset S \smallsetminus D_R(x)$. Moreover, $xw_{\circ,J}W_J$ is a facial $\equiv$-singleton for any $J \subseteq D_R(x)$.

Proof. (i) is an immediate consequence of Proposition (2.3.15). To prove (ii), we just need to show that if x is a $\equiv$-singleton, then $\pi_{\downarrow}(xw_{\circ,I}) = xw_{\circ,I}$ for any $I \subseteq S \smallsetminus D_R(x)$. If not, there would exist $t \in S$ such that $xtw_{\circ,I} \lneqq xw_{\circ,I}$. If $t \in I$, then $x \equiv xt$ by Lemma (2.3.3). If $t \notin I$, then $xt \leq x$. Since $\equiv$ is a lattice congruence and $xtw_{\circ,I} \equiv xw_{\circ,I}$,

$$x = x \wedge xw_{\circ,I} \equiv x \wedge xtw_{\circ,I} = xt.$$

In both cases, we contradict the assumption that x is a $\equiv$-singleton. We prove similarly that if x is a $\equiv$-singleton, then $\pi^{\uparrow}(xw_{\circ,J}) = xw_{\circ,J}$ for any $J \subseteq D_R(x)$. □

Remark 2.3.17 Proposition (2.3.16) (ii) can be interpreted as follows: if the weak order minimum or maximum element of a coset xW_I is a $\equiv$-singleton, then the coset xW_I is a facial $\equiv$-singleton. In fact, we conjecture that a coset is a facial $\equiv$-singleton if it contains a $\equiv$-singleton.

2.3.5 Root and weight inversion sets for facial congruence classes

As in Section (2.2.2), we now introduce and study the root and weight inversion sets of the congruence classes of $\equiv$. Root inversion sets are then used to obtain equivalent characterizations of the quotient lattice of the facial weak order by $\equiv$. Weight inversion sets are used later in Section (2.3.6) to describe all faces of N. Reading's fan $\mathcal{F}_\equiv$ associated to $\equiv$.

Definition 2.3.18 The *root inversion set* $\mathbf{R}(\Gamma)$ and the *weight inversion set* $\mathbf{W}(\Gamma)$ of a congruence class Γ of $\equiv$ are defined by

$$\mathbf{R}(\Gamma) = \bigcap_{zW_K \in \Gamma} \mathbf{R}(zW_K) \qquad \text{and} \qquad \mathbf{W}(\Gamma) = \bigcup_{zW_K \in \Gamma} \mathbf{W}(zW_K).$$

Proposition 2.3.19 *Consider a congruence class $\Gamma = [xW_I, yW_J]$ of $\equiv$.*

(i) The cones generated by the root and weight inversion sets of Γ are polar to each other:

$$\operatorname{cone}(\mathbf{R}(\Gamma))^\diamond = \operatorname{cone}(\mathbf{W}(\Gamma)).$$

(ii) The positive and negative parts of the root inversion set of Γ coincide with that of xW_I and yW_J:

$$\mathbf{R}(\Gamma) \cap \Phi^+ = \mathbf{R}(xW_I) \cap \Phi^+ \quad \textit{and} \quad \mathbf{R}(\Gamma) \cap \Phi^- = \mathbf{R}(yW_J) \cap \Phi^-.$$

(iii) The root and weight inversion sets of Γ can be computed from those of xW_I and yW_J by

$$\mathbf{R}(\Gamma) = \mathbf{R}(xW_I) \cap \mathbf{R}(yW_J) \quad \textit{and} \quad \mathbf{W}(\Gamma) = \mathbf{W}(xW_I) \cup \mathbf{W}(yW_J).$$

Proof. Since the polar of a union is the intersection of the polars, (i) is a direct consequence of Proposition (2.2.7) (iii).

For (ii), consider $zW_K \in \Gamma$. Since $xW_I \leq zW_K \leq yW_J$, we have by Remark (2.2.15)

$$\mathbf{R}(xW_I) \cap \Phi^+ \subseteq \mathbf{R}(zW_I) \cap \Phi^+ \quad \text{and} \quad \mathbf{R}(zW_K) \cap \Phi^- \supseteq \mathbf{R}(yW_J) \cap \Phi^-.$$

Therefore,

$$\begin{aligned} & \mathbf{R}(\Gamma) \cap \Phi^+ = \bigcap_{zW_K \in \Gamma} \mathbf{R}(zW_I) \cap \Phi^+ = \mathbf{R}(xW_I) \cap \Phi^+, \\ \text{and} \qquad & \mathbf{R}(\Gamma) \cap \Phi^- = \bigcap_{zW_K \in \Gamma} \mathbf{R}(zW_I) \cap \Phi^- = \mathbf{R}(yW_J) \cap \Phi^-. \end{aligned}$$

Finally, for (iii), we have already $\mathbf{R}(\Gamma) \subseteq \mathbf{R}(xW_I) \cap \mathbf{R}(yW_J)$. For the other inclusion, we have

$$\begin{aligned} & \mathbf{R}(xW_I) \cap \mathbf{R}(yW_J) \cap \Phi^+ \subseteq \mathbf{R}(xW_I) \cap \Phi^+ = \mathbf{R}(\Gamma) \cap \Phi^+ \subseteq \mathbf{R}(\Gamma), \\ \text{and} \qquad & \mathbf{R}(xW_I) \cap \mathbf{R}(yW_J) \cap \Phi^- \subseteq \mathbf{R}(yW_J) \cap \Phi^- = \mathbf{R}(\Gamma) \cap \Phi^- \subseteq \mathbf{R}(\Gamma). \end{aligned}$$

The equality on weights then follows by polarity. □

The following theorem is an analogue of Theorem (2.2.14). It provides characterizations of the quotient lattice of the facial weak order by $\equiv$ in terms of root inversion sets of the congruence classes, and of comparisons of the minimal and maximal elements in the congruence classes.

Theorem 2.3.20 *The following assertions are equivalent for two congruence classes* $\Gamma = [xW_I, yW_J]$ *and* $\Gamma' = [x'W_{I'}, y'W_{J'}]$ *of* $\equiv$*:*

(i) $\Gamma \leq \Gamma'$ *in the quotient of the facial weak order by* $\equiv$*,*

(ii) $xW_I \leq x'W_{I'}$*,*

(iii) $yW_J \leq y'W_{J'}$,

(iv) $xW_I \leq y'W_{J'}$,

(v) $x \leq y'$ *and* $xw_{\circ,I} \leq y'w_{\circ,J'}$,

(vi) $\mathbf{R}(\Gamma) \smallsetminus \mathbf{R}(\Gamma') \subseteq \Phi^-$ *and* $\mathbf{R}(\Gamma') \smallsetminus \mathbf{R}(\Gamma) \subseteq \Phi^+$,

(vii) $\mathbf{R}(\Gamma) \cap \Phi^+ \subseteq \mathbf{R}(\Gamma') \cap \Phi^+$ *and* $\mathbf{R}(\Gamma) \cap \Phi^- \supseteq \mathbf{R}(\Gamma') \cap \Phi^-$.

Proof. By definition, we have $\Gamma \leq \Gamma'$ in the quotient lattice if and only if there exists $zW_K \in \Gamma$ and $w_{K'} \in \Gamma'$ such that $zW_K \leq z'W_{K'}$. Therefore, any of Conditions (ii), (iii), and (iv) implies (i). Reciprocally, since $\Pi_\downarrow(zW_K) = xW_I$ and $\Pi_\downarrow(z'W_{K'}) = x'W_{I'}$, and $\Pi_\downarrow$ is order preserving, we get that (i) implies (ii). Similarly, since $\Pi^\uparrow(zW_K) = yW_J$, $\Pi^\uparrow(z'W_{K'}) = y'W_{J'}$, and $\Pi^\uparrow$ is order preserving, we get that (i) implies (iii). Since $xW_I \leq yW_J$ and $x'W_{I'} \leq y'W_{J'}$, either of (ii) and (iii) implies (iv). Moreover (iv) $\iff$ (v) by Theorem (2.2.14). We thus already obtained that (i) $\iff$ (ii) $\iff$ (iii) $\iff$ (iv) $\iff$ (v).

We now prove that (i) $\iff$ (vii). Assume first that $\Gamma \leq \Gamma'$. Since (i) implies (ii) and (iii), we have $xW_I \leq x'W_{I'}$ and $yW_J \leq y'W_{J'}$. By Remark (2.2.15) and Proposition (2.3.19) (ii), we obtain

$$\mathbf{R}(\Gamma) \cap \Phi^+ = \mathbf{R}(xW_I) \cap \Phi^+ \subseteq \mathbf{R}(x'W_{I'}) \cap \Phi^+ = \mathbf{R}(\Gamma') \cap \Phi^+,$$
$$\text{and} \qquad \mathbf{R}(\Gamma) \cap \Phi^- = \mathbf{R}(yW_J) \cap \Phi^- \supseteq \mathbf{R}(y'W_{J'}) \cap \Phi^- = \mathbf{R}(\Gamma') \cap \Phi^-.$$

Reciprocally, assume that (vii) holds. By Proposition (2.3.19) (ii), we have

$$\mathbf{R}(xW_I) \cap \Phi^+ \subseteq \mathbf{R}(x'W_{I'}) \cap \Phi^+ \quad \text{and} \quad \mathbf{R}(yW_J) \cap \Phi^- \supseteq \mathbf{R}(y'W_{J'}) \cap \Phi^-.$$

Since $xW_I \leq yW_J$ and $x'W_{I'} \leq y'W_{J'}$, we obtain by Remark (2.2.15) that

$$\mathbf{R}(xW_I) \cap \Phi^+ \subseteq \mathbf{R}(x'W_{I'}) \cap \Phi^+ \subseteq \mathbf{R}(y'W_{J'}) \cap \Phi^+$$
$$\text{and} \qquad \mathbf{R}(xW_I) \cap \Phi^- \subseteq \mathbf{R}(yW_J) \cap \Phi^- \supseteq \mathbf{R}(y'W_{J'}) \cap \Phi^-.$$

Again by Remark (2.2.15), we obtain that $xW_I \leq y'W_{J'}$, and thus that $\Gamma \leq \Gamma'$ since (iv) implies (i). This proves that (i) $\Longleftrightarrow$ (vii).

This conclude the proof as the equivalence (vii) $\Longleftrightarrow$ (vi) is immediate. □

2.3.6 Congruences and fans

Consider a lattice congruence $\equiv$ of the weak order and the corresponding congruence $\equiv$ of the facial weak order. N. Reading proved in (Reading, 2005, Proposition 5.2) that $\equiv$ naturally defines a complete fan which coarsens the Coxeter fan. Namely, for each congruence class γ of $\equiv$, consider the cone C_γ obtained by gluing the maximal chambers $\mathrm{cone}(x(\nabla))$ of the Coxeter fan corresponding to the elements x in γ. It turns out that each of these cones C_γ is convex and that the collection of cones $\{C_\gamma \mid \gamma \in W/{\equiv}\}$, together with all their faces, form a complete fan which we denote by $\mathcal{F}_\equiv$.

We now use the congruence $\equiv$ of the facial weak order to describe all cones of $\mathcal{F}_\equiv$ (not only the maximal ones). This shows that the lattice structure on the maximal faces of $\mathcal{F}_\equiv$ extends to a lattice structure on all faces of the fan $\mathcal{F}_\equiv$. Our description relies on the weight inversion sets defined in the previous section.

Proposition 2.3.21 *For a congruence class γ of $\equiv$ and the corresponding congruence class Γ of $\equiv$ such that $\gamma = W \cap \Gamma$ (see Corollary (2.3.13)), we have*

$$C_\gamma = \bigcup_{x \in \gamma} \mathrm{cone}(x(\nabla)) = \mathrm{cone}(\mathbf{W}(\Gamma)).$$

Proof. We have

$$C_\gamma = \bigcup_{x \in \gamma} \mathrm{cone}(x(\nabla)) = \bigcup_{x \in W \cap \Gamma} \mathrm{cone}(\mathbf{W}(x)) = \bigcup_{xW_I \in \Gamma} \mathrm{cone}(\mathbf{W}(xW_I)) = \mathrm{cone}(\mathbf{W}(\Gamma)).$$

□

Theorem 2.3.22 *The collection of cones* $\{\mathrm{cone}(\mathbf{W}(\Gamma)) \mid \Gamma \in \mathcal{P}_W/{\equiv}\}$ *forms the complete fan* $\mathcal{F}_{\equiv}$*.*

Proof. Denote by $\mathcal{C}$ the collection of cones $\{\mathrm{cone}(\mathbf{W}(\Gamma)) \mid \Gamma \in \mathcal{P}_W/{\equiv}\}$. The relative interiors of the cones of $\mathcal{C}$ form a partition of the ambient space V, since $\equiv$ is a congruence of the Coxeter complex $\mathcal{P}_W$. Similarly, the relative interiors of the cones of $\mathcal{F}_{\equiv}$ form a partition of the ambient space V since we already know that $\mathcal{F}_{\equiv}$ is a complete fan (Reading, 2005). Therefore, we only have to prove that each cone of $\mathcal{F}_{\equiv}$ is a cone of $\mathcal{C}$. First, Proposition (2.3.21) ensures that the full-dimensional cones of $\mathcal{C}$ are precisely the full-dimensional cones of $\mathcal{F}_{\equiv}$. Consider now another cone F of $\mathcal{F}_{\equiv}$, and let C and C' be the minimal and maximal full-dimensional cones of $\mathcal{F}_{\equiv}$ containing F (in the order given by $\leq /{\equiv}$). Since C and C' are full-dimensional cones of $\mathcal{F}_{\equiv}$, there exist congruence classes Γ and Γ' of $\equiv$ such that $C = \mathrm{cone}(\mathbf{W}(\Gamma))$ and $C' = \mathrm{cone}(\mathbf{W}(\Gamma'))$. One easily checks that the Coxeter cones contained in the relative interior of F are precisely the cones $\mathrm{cone}(\mathbf{W}(xW_I))$ for the cosets xW_I such that $x \in \Gamma$ while $xw_{\circ,I} \in \Gamma'$. By Proposition (2.3.12), these cosets form a congruence class Ω of $\equiv$. It follows that $F = \mathrm{cone}(\mathbf{W}(\Omega)) \in \mathcal{C}$, thus concluding the proof. □

Corollary 2.3.23 *A coset* xW_I *is a facial* $\equiv$*-singleton if an only if* $\mathrm{cone}(\mathbf{W}(xW_I))$ *is a cone of* $\mathcal{F}_{\equiv}$*.*

2.3.7 Two examples: Facial boolean and Cambrian lattices

To illustrate the results in this section, we revisit two relevant families of lattice congruences of the weak order, namely the descent congruence and the Cambrian congruences (Reading, 2006).

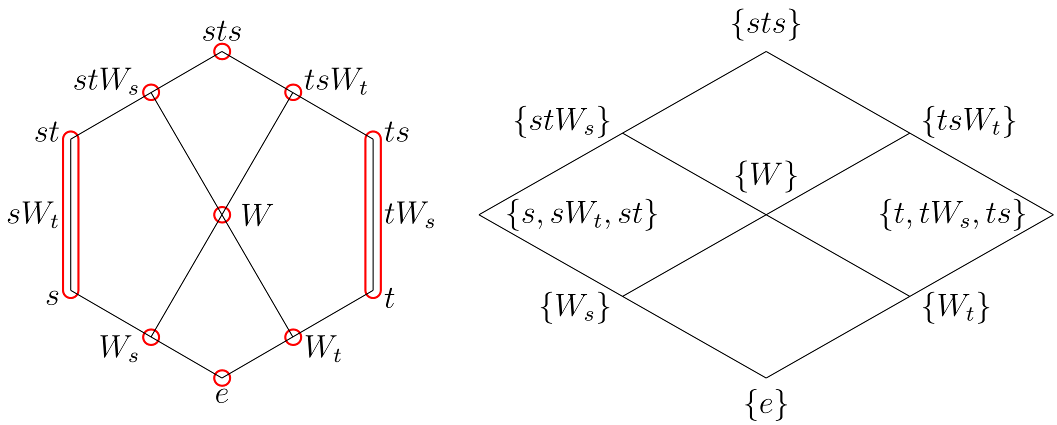


Figure 2.9: The descent congruence classes of the standard parabolic cosets in type A_2 (left) and the resulting quotient (right).

Facial boolean lattices

The *descent congruence* is the congruence of the weak order defined by $x \equiv^{\mathsf{des}} y$ if and only if $D_L(x) = D_L(y)$. The corresponding up and down projections are given by $\pi_{\downarrow}(x) = w_{\circ, D_L(x)}$ and $\pi^{\uparrow}(x) = w_\circ w_{\circ, S \smallsetminus D_L(x)}$. The quotient of the weak order by $\equiv^{\mathsf{des}}$ is isomorphic to the boolean lattice on S. The fan $\mathcal{F}_{\equiv}^{\mathsf{des}}$ is given by the arrangement of the hyperplanes orthogonal to the simple roots of Δ. It is the normal fan of the parallelepiped $\mathsf{Para}(W)$ generated by the simple roots of Δ.

Denote by $\boldsymbol{\equiv}^{\mathsf{des}}$ the facial weak order congruence induced by $\equiv^{\mathsf{des}}$ as defined in Section (2.3.3). According to Theorem (2.3.22), the $\boldsymbol{\equiv}^{\mathsf{des}}$ congruence classes correspond to all faces of the parallelepiped $\mathsf{Para}(W)$.

In the next few statements, we provide a direct criterion to test whether two cosets are $\boldsymbol{\equiv}^{\mathsf{des}}$-congruent. For this, we need to extend to all cosets the notion of descent sets.

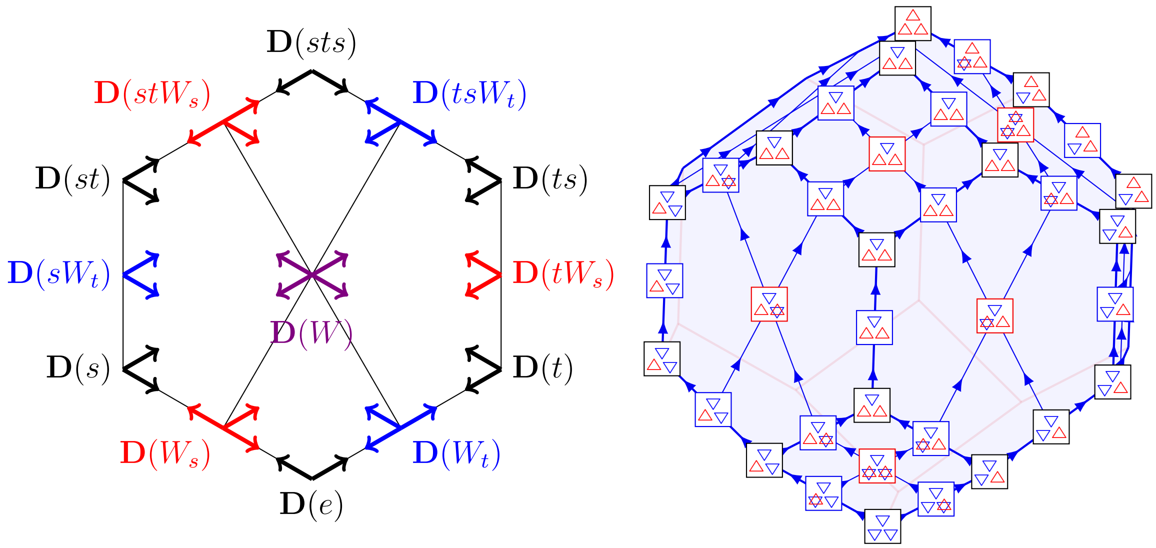


Figure 2.10: The root descent sets of the standard parabolic cosets in type A_2 (left) and A_3 (right).

Definition 2.3.24 Let the (left) *root descent set* of a coset xW_I be the set of roots

$$\mathbf{D}(xW_I) \coloneqq \mathbf{R}(xW_I) \cap \pm\Delta \ \subseteq \ \Phi.$$

Figure (2.10) illustrates the root descent sets in type A_2 (left) and A_3 (right). For the latter, we have just discarded the interior triangles in each root inversion set in Figure (2.7).

Notice that the simple roots in the inversion set $\mathbf{N}(x)$ precisely correspond to the descent set $D_L(x)$:

$$\Delta \cap \mathbf{N}(x) = \{\alpha_s \mid \ s \in D_L(x)\} = \{\alpha_s \mid \ s \in S,\, \ell(sx) < \ell(x)\}\,.$$

Similar to Proposition (2.2.10), the next statement concerns the root descent set $\mathbf{D}(xW_\varnothing)$ for $x \in W$. For brevity we write $\mathbf{D}(x)$ instead of $\mathbf{D}(xW_\varnothing)$.

Proposition 2.3.25 *For any $x \in W$, the root descent set $\mathbf{D}(x)$ has the following properties.*

(i) $\mathbf{D}(x) = \Big(\Delta \cap \mathbf{N}(x)\Big) \cup -(\Delta \smallsetminus \mathbf{N}(x))$. *In other words,*

$$\mathbf{D}(x) \cap \Phi^+ = \Big(\Delta \cap \mathbf{N}(x)\Big) \qquad \textit{and} \qquad \mathbf{D}(x) \cap \Phi^- = -\Big(\Delta \smallsetminus \mathbf{N}(x)\Big).$$

(ii) $\mathbf{D}(xw_\circ) = -\mathbf{D}(x)$ *and* $\mathbf{D}(w_\circ x) = w_\circ\Big(\mathbf{D}(x)\Big)$.

Proof. The results follow immediately from Proposition (2.2.10) by intersecting with $\pm\Delta$ appropriately. □

As in Proposition (2.2.12) and Corollary (2.2.13), the root descent set of a coset xW_I can be computed from that of its minimal and maximal length representatives x and $xw_{\circ,I}$.

Proposition 2.3.26 *The root and weight inversion sets of xW_I can be computed from those of x and $xw_{\circ,I}$ by $\mathbf{D}(xW_I) = \mathbf{D}(x) \cup \mathbf{D}(xw_{\circ,I})$. In other words,*

$$\mathbf{D}(xW_I) \cap \Phi^- = \mathbf{D}(x) \cap \Phi^- \qquad \textit{and} \qquad \mathbf{D}(xW_I) \cap \Phi^+ = \mathbf{D}(xw_{\circ,I}) \cap \Phi^+.$$

Proof. Follow immediately from Proposition (2.2.12) and Corollary (2.2.13) by intersecting with $\pm\Delta$ appropriately. □

From the previous propositions, we obtain that the $\equiv^{\mathsf{des}}$-equivalence class of xW_I is determined by the root descent set $\mathbf{D}(xW_I)$.

Proposition 2.3.27 *For any cosets xW_I, yW_J, we have $xW_I \equiv^{\mathsf{des}} yW_J$ if and only if $\mathbf{D}(xW_I) = \mathbf{D}(yW_J)$.*

Proof. As observed in Proposition (2.3.12), the $\equiv^{\mathsf{des}}$-congruence class of xW_I only depends on the $\equiv^{\mathsf{des}}$-congruence class of x and $xw_{\circ,I}$, and thus on the descent sets $D_L(x)$ and $D_L(xw_{\circ,I})$. By Propositions (2.3.25) (i) and (2.3.26), the root descent set $\mathbf{D}(xW_I)$ and the descent sets $D_L(x)$ and $D_L(xw_{\circ,I})$ determine each other. We conclude that the $\equiv^{\mathsf{des}}$-equivalence class of xW_I is determined by the root descent set $\mathbf{D}(xW_I)$. □

Finally, we observe that the facial $\equiv^{\mathsf{des}}$-singletons correspond to the bottom and top faces of the W-permutahedron.

Proposition 2.3.28 *A coset xW_I is a facial $\equiv^{\mathsf{des}}$-singleton if and only if $x = e$ or $xw_{\circ,I} = w_\circ$.*

Proof. As already mentioned, the up and down projection maps of the descent congruence are given by $\pi_\downarrow(x) = w_{\circ,D_L(x)}$ and $\pi^\uparrow(x) = w_\circ w_{\circ,S \smallsetminus D_L(x)}$. From Proposition (2.3.16), we therefore obtain that a coset xW_I is a singleton if and only if $w_\circ w_{\circ,S \smallsetminus D_L(x)} = x$ and $w_{\circ,D_L(xw_{\circ,I})} = xw_{\circ,I}$. The result follows. □

Example 2.3.29 In type A, the *descent vector* of an ordered partition λ of $[n]$ is the vector $\operatorname{des}(\lambda) \in \{-1, 0, 1\}^{n-1}$ given by

$$\operatorname{des}(\lambda)_i = \begin{cases} -1 & \text{if } \lambda^{-1}(i) < \lambda^{-1}(i+1), \\ 0 & \text{if } \lambda^{-1}(i) = \lambda^{-1}(i+1), \\ 1 & \text{if } \lambda^{-1}(i) > \lambda^{-1}(i+1). \end{cases}$$

These descent vectors where used by J.-C. Novelli and J.-Y. Thibon in (Novelli & Thibon, 2006) to see that the facial weak order on the cube is a lattice. See also (Chatel & Pilaud, 2014).

Facial Cambrian lattices

Fix a *Coxeter element* c, *i.e.*, the product of all simple reflections in S in an arbitrary order. A simple reflection $s \in S$ is *initial* in c if $\ell(sc) < \ell(c)$. For s

initial in c, note that scs is another Coxeter element for W while sc is a Coxeter element for $W_{S \smallsetminus \{s\}}$.

In (Reading, 2006; Reading, 2007b), N. Reading defines the c-Cambrian lattice as a lattice quotient of the weak order (by the c-Cambrian congruence) or as a sublattice of the weak order (induced by c-sortable elements). There are several ways to present his constructions, we choose to start from the projection maps of the c-Cambrian congruence (as we did in the previous sections). These maps are defined by an induction both on the length of the elements and on the rank of the underlying Coxeter group. Namely, define the projection $\pi_\downarrow^c : W \to W$ inductively by $\pi_\downarrow^c(e) = e$ and for any s initial in c,

$$\pi_\downarrow^c(w) = \begin{cases} s \cdot \pi_\downarrow^{scs}(sw) & \text{if } \ell(sw) < \ell(w) \\ \pi_\downarrow^{sc}(w_{\langle s \rangle}) & \text{if } \ell(sw) > \ell(w), \end{cases}$$

where $w = w_{\langle s \rangle} \cdot {}^{\langle s \rangle}w$ is the unique factorization of w such that $w_{\langle s \rangle} \in W_{S \smallsetminus \{s\}}$ and $\ell(t^{\langle s \rangle}w) > \ell({}^{\langle s \rangle}w)$ for all $t \in S \smallsetminus \{s\}$. The projection $\pi_c^\uparrow : W \to W$ can then be defined similarly, or by

$$\pi_c^\uparrow(w) = \left(\pi_\downarrow^{(c^{-1})}(ww_\circ)\right)w_\circ.$$

N. Reading proves in (Reading, 2007b) that these projection maps $\pi_c^\uparrow$ and $\pi_\downarrow^c$ satisfy the properties of Lemma (2.3.2) and therefore define a congruence $\equiv^c$ of the weak order called *c-Cambrian congruence*. The quotient of the weak order by the c-Cambrian congruence is called the *c-Cambrian lattice*. It was also defined as the smallest congruence contracting certain edges, see (Reading, 2006).

Cambrian congruences are relevant in the context of finite type cluster algebras, generalized associahedra, and W-Catalan combinatorics. Without details, let us point out the following facts:

(i) The fan $\mathcal{F}_{\equiv^c}$ associated to the c-Cambrian congruence $\equiv^c$ is the *Cambrian* *fan* studied by N. Reading and D. Speyer (Reading & Speyer, 2009). Is was proved to be the normal fan of a polytope by C. Hohlweg, C. Lange and H. Thomas (Hohlweg et al., 2011). See also (Stella, 2013; Pilaud & Stump, 2015) for further geometric properties. The resulting polytopes are called *generalized associahedra*.

(ii) These polytopes realize the *c-cluster complexes* of type W. When W is crystallographic, these complexes were defined from the theory of finite type *cluster algebras* of S. Fomin and A. Zelevinsky (Fomin & Zelevinsky, 2002; Fomin & Zelevinsky, 2003).

(iii) The minimal elements in the c-Cambrian congruence classes are precisely the *c-sortable elements*, defined as the elements $w \in W$ such that there exists nested subsets $K_1 \supseteq K_2 \supseteq \cdots \supseteq K_r$ of S such that $w = c_{K_1} c_{K_2} \dots c_{K_r}$ where c_K is the product of the elements in K in the order given by c. The maximal elements of the c-Cambrian congruence classes are the *c-antisortable elements*, defined as the elements $w \in W$ such that $ww_\circ$ is c^{-1}-sortable. N. Reading proved in (Reading, 2007b) that the Cambrian lattice is in fact isomorphic to the sublattice of the weak order induced by c-sortable elements (or by c-antisortable elements). The c-sortable elements are connected to various W-Catalan families: c-clusters, vertices of the c-associahedron, W-non-crossing partitions. See (Reading, 2007a) for precise definitions.

The results presented in this paper translate to the following statement.

Theorem 2.3.30 *For any Coxeter element c of W, the facial c-Cambrian congruence $\equiv^c$ on the Coxeter complex $\mathcal{P}_W$, defined by*

$$xW_I \equiv^c yW_J \iff x \equiv^c y \text{ and } xw_{\circ,I} \equiv^c yw_{\circ,J},$$

has the following properties:

(i) The c-Cambrian congruence $\equiv^c$ *is the restriction of the facial c-Cambrian congruence* $\overline{\equiv}^c$ *to* W.

(ii) The quotient of the facial weak order by the facial c-Cambrian congruence $\overline{\equiv}^c$ *defines a lattice structure on the cones of the c-Cambrian fan of (Reading & Speyer, 2009), or equivalently on the faces of the c-associahedron of (Hohlweg et al., 2011).*

(iii) A coset xW_I *is minimal (resp. maximal) in its facial c-congruence class if and only if* $xw_{\circ,I}$ *is c-sortable (resp.* x *is c-antisortable). In particular, a Coxeter cone* $\mathrm{cone}(\mathbf{W}(xW_I))$ *is a cone of the c-Cambrian fan if and only if* x *is c-antisortable and* $xw_{\circ,I}$ *is c-sortable.*

Proof. (i) is an application of Corollary (2.3.13). (iii) follows from Theorem (2.3.22) and the fact that the c-Cambrian fan of (Reading & Speyer, 2009) is the normal fan of the c-associahedron of (Hohlweg et al., 2011). Finally, (iii) is a direct translation of Propositions (2.3.15) and (2.3.16). □

Example 2.3.31 Examples of facial Cambrian congruences in type A_2, A_3, and B_3 are represented in Figures (2.11), (2.12), and (2.13) respectively.

Example 2.3.32 In type A, the Tamari congruence classes correspond to binary trees, while the facial Tamari congruence classes correspond to Schröder trees. The quotient of the facial weak order by the facial Tamari congruence was already described in (Palacios & Ronco, 2006; Novelli & Thibon, 2006). In (Chatel & Pilaud, 2014), G. Chatel and V. Pilaud describe the Cambrian counterparts of binary trees and Schröder trees, and use them to introduce the facial type A Cambrian lattices.

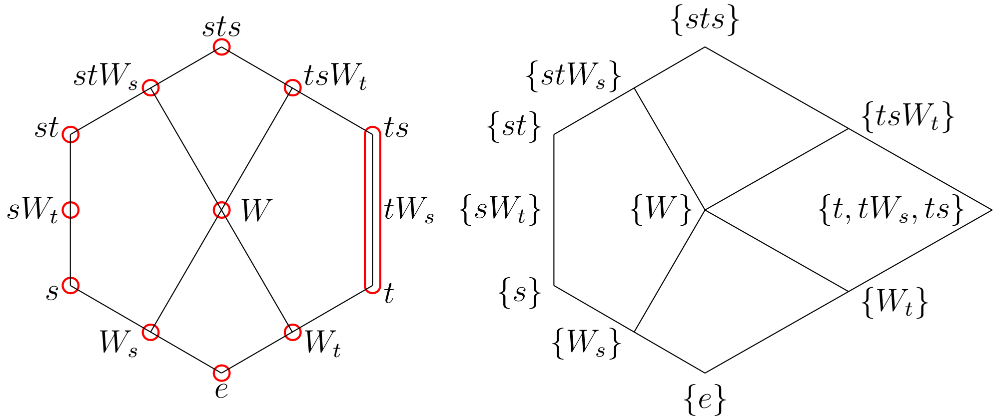


Figure 2.11: The st-Cambrian congruence classes of the standard parabolic cosets in type A_2 (left) and the resulting quotient (right).

Remark 2.3.33 If $\equiv$ is an order congruence on a poset $(P, \leq)$ with up and down projections $\pi^{\uparrow}$ and $\pi_{\downarrow}$, the suborder of $\leq$ induced by $\pi_{\downarrow}(P)$ is isomorphic to the quotient order $P/{\equiv}$ (see Definition (2.3.1)). When P is a lattice, $P/{\equiv}$ is also a lattice, so that $(\pi_{\downarrow}(P), \leq)$ is a lattice. Although $(\pi_{\downarrow}(P), \leq)$ is always a meet subsemilattice of P, it is not necessarily a sublattice of P. In (Reading, 2007b), N. Reading proved moreover that the weak order induced on $\pi_{\downarrow}(W)$ for the Cambrian congruence is actually a sublattice of the weak order on W. In contrast, the facial weak order induced on $\Pi_{\downarrow}(\mathcal{P}_W)$ is not a sublattice of the facial weak order on $\mathcal{P}_W$. An example already appears in A_3 for $c = srt$. Consider $xW_I = tsrW_{st}$ and $yW_J = stsrW_s$, so that $xW_I \wedge yW_J = z_{\wedge} W_{K_{\wedge}} = tsrW_t$. We observe that $xw_{\circ,I} = srt|srt = w_{\circ}$ and $yw_{\circ,J} = srt|sr$ are srt-sortable while $z_{\wedge} w_{\circ,K_{\wedge}} = st|sr$ is not.

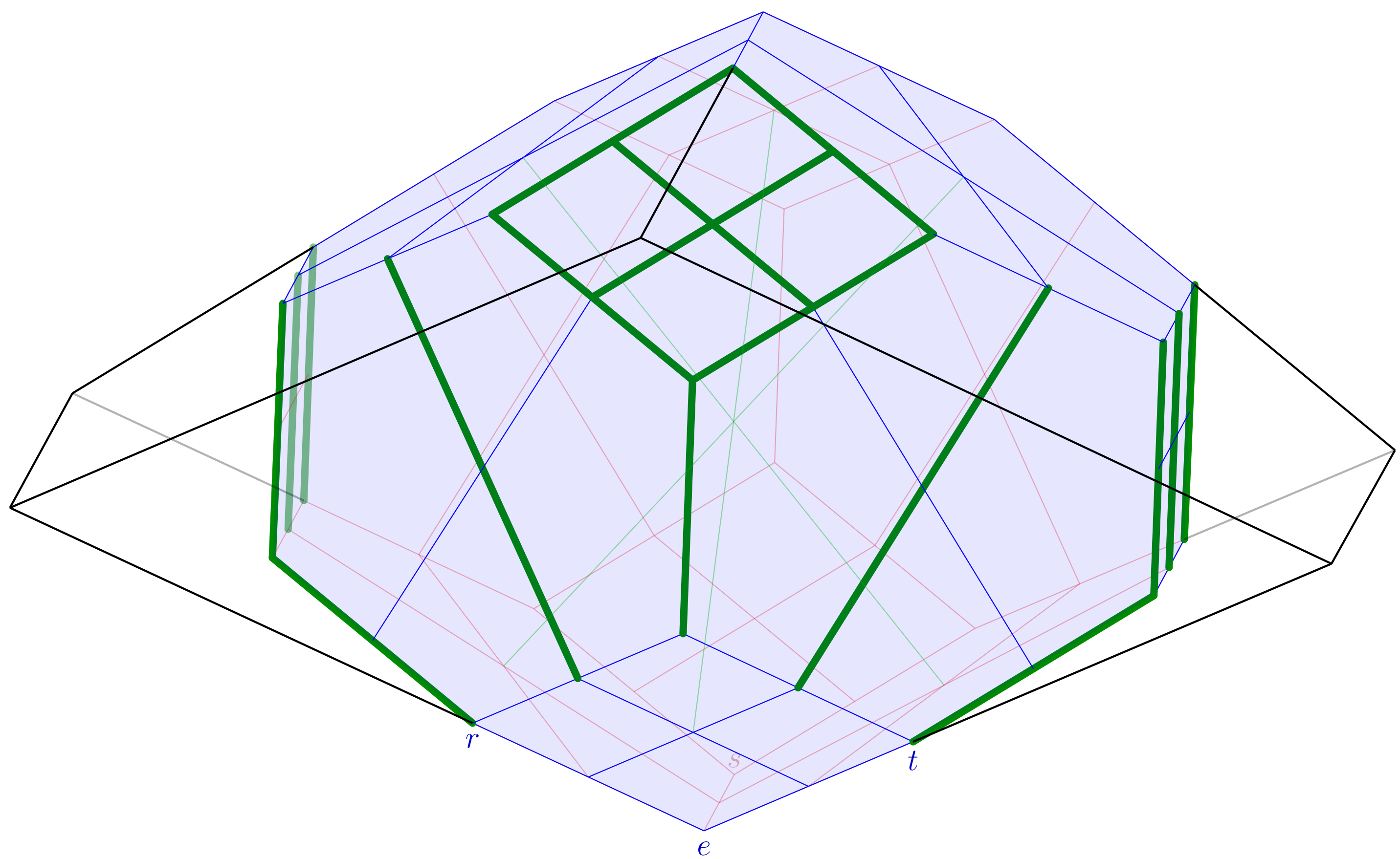


Figure 2.12: The srt-Cambrian congruence on the standard parabolic cosets in type A_3 and the resulting quotient. Bold green edges are contracted edges. The quotient is given the geometry of the type A_3 srt-associahedron of (Hohlweg et al., 2011).

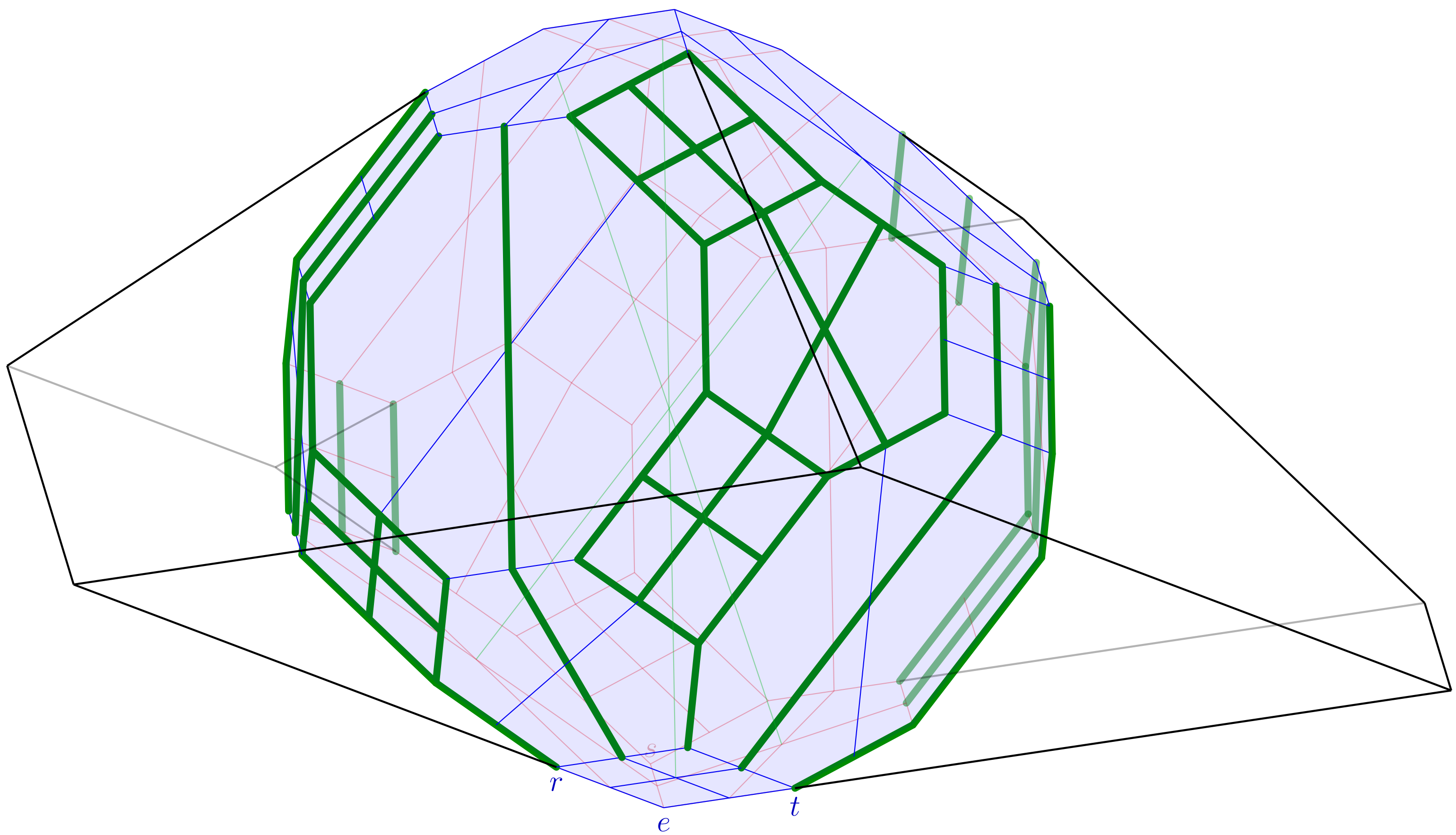


Figure 2.13: The srt-Cambrian congruence on the standard parabolic cosets in type B_3 and the resulting quotient. Bold green edges are contracted edges. The quotient is given the geometry of the type B_3 srt-associahedron of (Hohlweg et al., 2011).

CHAPTER III

HYPERPLANE ARRANGEMENTS AND ORIENTED MATROIDS

In this chapter we prepare the reader for our second article (Dermenjian et al., 2019), which we present in Chapter 4. We start with an introduction to hyperplane arrangements in § 3.1. In § 3.2 we survey the regions and faces of a hyperplane arrangement and the face lattice of an arrangement. We then discuss in § 3.3 the notion of an associated essential arrangement of an arrangement by decreasing the dimension of the vector space without altering the structure of the face lattice, thus removing unnecessary information from our arrangements. We then describe another poset structure called the poset of regions on a hyperplane arrangement in § 3.4, which is a generalization of the weak order on Coxeter groups through the use of Coxeter arrangements. The poset of regions is the poset we extend to the facial weak order on hyperplane arrangements in Chapter 4. In § 3.5 we define the notion of simplicial arrangements, a large family of hyperplane arrangements whose poset of regions is a lattice, which contain the Coxeter arrangements as a subfamily. We then survey covectors in § 3.6, a representation of the faces of a hyperplane arrangement using sign vectors relative to the normal vectors chosen of our hyperplanes, allowing us to use algebraic techniques when working with faces of an arrangement. In § 3.7 we survey four operations on covectors and their interpretations in hyperplane arrangements. Finally, we survey oriented matroids, a generalization of hyperplane arrangements, in § 3.8 using the covector axioms.

The covector terminology and the generalization to oriented matroids will help facilitate the proofs in Chapter 4. As with Chapter 1, no proofs are contained in this chapter. For a more thorough background on the topics covered in this chapter and for proofs, the reader is invited to consult the book “Arrangements of Hyperplanes” by Orlik and Terao (Orlik & Terao, 1992), the book “Lectures on Polytopes” by Ziegler (Ziegler, 1995), and the book “Oriented Matroids” by Björner, Las Vergnas, Sturmfels, White and Ziegler (Björner et al., 1999).

3.1 Hyperplane arrangements

Let $(V, \langle \cdot, \cdot \rangle)$ be an n-dimensional real Euclidean vector space. A *hyperplane* is a codimension 1 (dimension $n-1$) linear subspace of V that separates V into two distinct regions. A *(central) hyperplane arrangement*, or *arrangement* for short, is a finite set $\mathcal{A}$ of hyperplanes in V which intersect the origin. As every hyperplane H separates V into two distinct regions, we distinguish the regions by first choosing some fixed nonzero vector e_H normal to H, *i.e.*, $H = \{v \in V \mid \langle e_H, v \rangle = 0\}$. Then $H^+ = \{v \in V \mid \langle e_H, v \rangle \geq 0\}$ is the *positive half-space of the hyperplane* H and $H^- = \{v \in V \mid \langle e_H, v \rangle \leq 0\}$ is the *negative half-space of* H. Note that the choice of vectors e_H are unique up to nonzero scalar multiplication.

Example 3.1.1 In Figure 3.1 are three examples of (central) hyperplane arrangements. The first two arrangements in Figure 3.1 live in $\mathbb{R}^2$ and the third arrangement in Figure 3.1 lives in $\mathbb{R}^3$.

In the first arrangement of Figure 3.1, a normal vector e to the only hyperplane H is given. The positive half-space of H is the set H^+ of points weakly above and to the left of the hyperplane. Similarly, the negative half-space H^- is the set of points weakly below and to the right of the hyperplane. Note that these half-spaces are dependent on the choice of e. If we had chosen $e' = -e$ as our normal

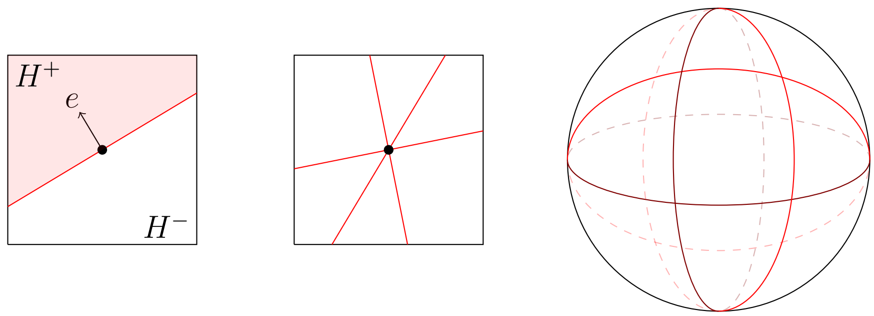


Figure 3.1: Two central hyperplane arrangements in a 2-dimensional real Euclidean vector space and a central hyperplane arrangement in a 3-dimensional real Euclidean vector space. The arrangement on the left is simplicial but not essential, the arrangement on the right is essential but not simplicial and the arrangement in the middle is both essential and simplicial.

vector, then our half-spaces H^+ and H^- would be exchanged.

A large family of hyperplane arrangements are given by Coxeter groups. Consider a finite Coxeter system (W, S) with root system Φ and positive roots Φ^+. Let $\mathcal{A}_W$ denote the hyperplane arrangement given by

$$\mathcal{A}_W := \left\{ H_\alpha \mid \alpha \in \Phi^+ \right\}.$$

The arrangement $\mathcal{A}_W$ is known as a *Coxeter arrangement*. The *type* of a Coxeter arrangement $\mathcal{A}_W$ is the type of its underlying Coxeter group W.

Example 3.1.2 Figure 3.2 gives an example of the Coxeter arrangement for the type A_2 Coxeter group where $S = \{s_{e_1}, s_{e_3}\}$, $\Delta = \{e_1, e_3\}$ and $\Phi^+ = \{e_1, e_2, e_3\}$.

The hyperplanes of an arrangement cut up the vector space into various connected components which we survey in the following section.

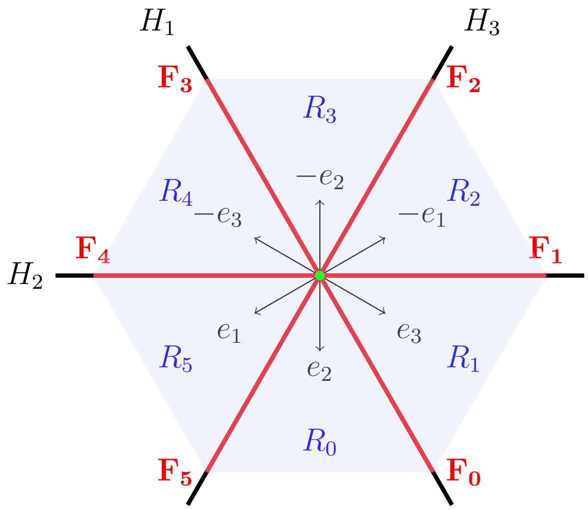


Figure 3.2: The type A_2 Coxeter arrangement where R_0 is the intersection of the positive half-spaces of all hyperplanes.

3.2 Regions, faces and the face lattice

In this section we survey the notions of regions and faces of an arrangement.

Let $\mathcal{A}$ be an arrangement in an n-dimensional real Euclidean vector space V. The *set of regions* $\mathscr{R}_\mathcal{A}$ *of* $\mathcal{A}$ consists of the connected components of $V \setminus (\cup_{H \in \mathcal{A}} H)$. In other words, the regions are the closures of the connected components left over when the hyperplanes are removed from the vector space. A *face of an arrangement* $\mathcal{A}$ is the intersection of the closures of some regions in $\mathscr{R}_\mathcal{A}$. Let $\mathscr{F}_\mathcal{A}$ denote the set of faces of $\mathcal{A}$, *i.e.*, $\mathscr{F}_\mathcal{A} = \{\bigcap_{R \in \mathcal{R}} R \mid \mathcal{R} \subseteq \mathscr{R}_\mathcal{A}\}$. Note that the codimension 0 faces of $\mathcal{A}$ are the regions.

Example 3.2.1 Let $\mathcal{A}$ be the Coxeter arrangement of type A_2 as in Figure 3.2. Removing the hyperplanes in $\mathcal{A}$ from the vectors space $\mathbb{R}^2$ we are left with six connected components. The closure of these six connected components are the regions (in blue)

$$\mathscr{R}_{\mathcal{A}} = \{R_0, R_1, R_2, R_3, R_4, R_5\}.$$

The faces $\mathscr{F}_{\mathcal{A}}$ of this arrangement are given by the intersections of the six regions in $\mathscr{R}_{\mathcal{A}}$. For example, the face F_0 is obtained by taking the intersection $R_0 \cap R_1$. By taking arbitrary intersections we observe that there are 13 faces in total: the closures of the six regions R_i, six codimension 1 faces denoted by the F_i, given in red, and the centre codimension 2 face, $\{0\}$, in green.

The faces of the hyperplane arrangement $\mathcal{A}$, together with the vector space V itself, ordered by inclusion is a poset called the *face lattice of* $\mathcal{A}$, denoted $(\mathscr{F}_{\mathcal{A}}, \subseteq)$. The bottom element of this lattice is given by $\{0\}$ since every face contains the origin and the top element of this lattice is the vector space itself. In general, when the arrangement is essential, the bottom element of $(\mathscr{F}_{\mathcal{A}}, \subseteq)$ is $\{0\}$, the top element is the vector space V, the atoms are the rays and the coatoms are the regions.

Example 3.2.2 Let $\mathcal{A}$, $\mathcal{A}'$, and $\mathcal{A}''$ be the three hyperplane arrangements in Figure 3.1 respectively. In Figure 3.3 we give the Hasse diagrams for the face lattice for each arrangement. The Hasse diagram for $(\mathscr{F}_{\mathcal{A}}, \subseteq)$ is in the top left, the Hasse diagram for $(\mathscr{F}_{\mathcal{A}'}, \subseteq)$ is in the top right and the Hasse diagram for $(\mathscr{F}_{\mathcal{A}''}, \subseteq)$ is on the bottom of Figure 3.3 respectively.

3.3 Essential arrangements

Although hyperplane arrangements can live in any dimensional real Euclidean vector space V, it is easier to work with arrangements that live in real Euclidean vector spaces that have minimal dimension. It turns out, as we will see in this section, we can decrease the dimension of the vector space V that an arrangement lives in without changing the structure of the face lattice.

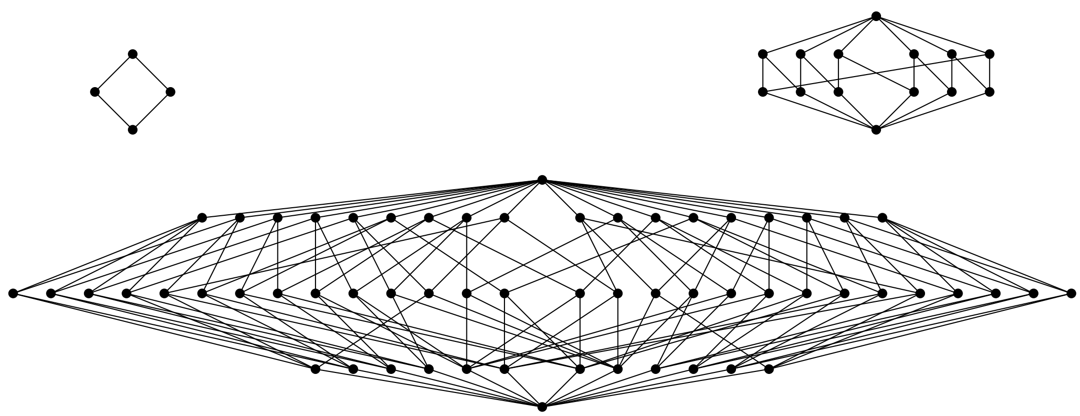


Figure 3.3: The top left graph is the Hasse diagram of the face lattice of the left-most hyperplane arrangement in Figure 3.1. The top right graph is the Hasse diagram of the face lattice of the middle hyperplane arrangement in Figure 3.1. Note that it is also the Hasse diagram of the face lattice of the Coxeter arrangement of type A_2 as in Figure 3.2. Finally, the bottom graph is the Hasse diagram of the face lattice of the right-most hyperplane arrangement in Figure 3.1.

The *rank* of an arrangement $\mathcal{A}$, denoted $\text{rank}(\mathcal{A})$, is the dimension of the linear subspace V' spanned by the normal vectors e_H for $H \in \mathcal{A}$. If $\text{rank}(\mathcal{A}) = \dim(V)$ then the arrangement is said to be *essential*. We assume all arrangements to be essential unless stated otherwise. We do not lose generality by restricting to essential arrangements since for every arrangement $\mathcal{A}$ with rank r living in $V \cong \mathbb{R}^n$ (with $r \leq n$) there is an *associated essential arrangement* $\mathcal{A}'$ with rank r living in $V' \cong \mathbb{R}^r$ such that $(\mathscr{F}_{\mathcal{A}}, \subseteq)$ and $(\mathscr{F}_{\mathcal{A}'}, \subseteq)$ are isomorphic, see the discussion in (Björner et al., 1999, Section 2.1).

Example 3.3.1 The first arrangement $\mathcal{A}$ in Figure 3.1 is not essential. There is only one hyperplane H in $\mathcal{A}$ whose normal vector e spans a 1-dimensional subspace of $\mathbb{R}^2$ implying the rank of the first arrangement is 1 while the arrangement itself lives in $\mathbb{R}^2$, a 2 dimensional space. Since $\text{rank}(\mathcal{A}) = 1 \neq 2 = \dim(V)$, this first arrangement is not essential. The associated essential arrangement of $\mathcal{A}$ is given by restricting our vector space to the space $V' = \{\lambda e \in V \mid \lambda \in \mathbb{R}\}$. This restriction produces an arrangement $\mathcal{A}'$, given in Figure 3.4, which also contains a single hyperplane and separates V' into two segments. The face lattice of $\mathcal{A}'$ is then isomorphic to the face lattice of $\mathcal{A}$ whose Hasse diagram is on the top left of Figure 3.3. The face lattice $(\mathscr{F}_{\mathcal{A}'}, \subseteq)$ has the point $\{0\}$ (the hyperplane) as the bottom element, the two line segments (two half-spaces) as the atoms, and the vector space (the line) V' as the top element.

On the other hand, the other two arrangements in Figure 3.1 are essential. These two arrangements live in $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively and, by taking any choice of normal vectors to the hyperplanes in each arrangement, the rank of the arrangements are 2 and 3 respectively. Therefore, since the ranks are equal to the dimensions of their respective vector spaces, they are both essential.

Having described a lattice on the faces of the hyperplane arrangement, we next

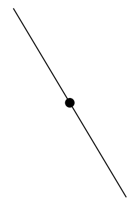


Figure 3.4: An associated essential hyperplane $\mathcal{A}'$ to the left-most hyperplane arrangement $\mathcal{A}$ in Figure 3.1.

describe another poset structure on hyperplane arrangements; this time using regions.

3.4 Poset of regions

In this section we survey a poset structure on the regions of an arrangement which is the generalization of the weak order on Coxeter groups. For a background on orders, posets or lattices the reader is referred to Appendix A where we have provided a brief introduction on order theory.

Let $\mathcal{A}$ be a hyperplane arrangement with set of regions $\mathscr{R}_{\mathcal{A}}$. The *separation set* $S(R, R')$ between two regions $R, R' \in \mathscr{R}_{\mathcal{A}}$ is the set of hyperplanes of $\mathcal{A}$ which separate the two regions:

$$S(R, R') := \{H \in \mathcal{A} \mid \ H \text{ separates } R \text{ from } R'\}.$$

Fix a region $B \in \mathscr{R}_{\mathcal{A}}$ and call it the *base region*. For brevity, we let $S(R) = S(B, R)$ be the separation set between a region R and the base region B. The *poset of regions* $\mathrm{PR}(\mathcal{A}, B)$ is the partial order on regions $\mathscr{R}_{\mathcal{A}}$ given by inclusion of separation sets $S(R)$, *i.e.*, for $R, R' \in \mathscr{R}_{\mathcal{A}}$:

$$R \leq_{\mathrm{PR}} R' \text{ if and only if } S(R) \subseteq S(R').$$

Example 3.4.1 Let W be a type A_2 Coxeter group with Coxeter arrangement as in Figure 3.2. We use Figure 3.2 to determine our separation sets. Taking R_1 and R_3, it can be observed that there are only two hyperplanes that are between these two regions, namely H_2 and H_3. Therefore $S(R_1, R_3) = \{H_2, H_3\}$.

For an example of a poset of regions, we fix a base region, say $B = R_0$, and determine the separation set $S(R)$ for each region $R \in \mathscr{R}_{\mathcal{A}}$. Considering $R_1 \in \mathscr{R}_{\mathcal{A}}$, the only hyperplane separating R_1 from $R_0 = B$ is the hyperplane H_1. Therefore the separation set between the region R_1 and the base region is the set $\{H_1\}$, in other words $S(R_1) = S(B, R_1) = \{H_1\}$. By a similar calculation, we have the following separation sets for each region:

$$\begin{array}{ccc} S(B) = \emptyset & S(R_1) = \{H_1\} & S(R_2) = \{H_1, H_2\} \\ S(R_3) = \{H_1, H_2, H_3\} = \mathcal{A} & S(R_4) = \{H_2, H_3\} & S(R_5) = \{H_3\}. \end{array}$$

Ordering the separation sets by inclusion gives us the poset of regions. The Hasse diagram of the poset of regions $\mathrm{PR}(\mathcal{A}, B)$ where $B = R_0$ is given in Figure 3.5. The Hasse diagram for this poset is isomorphic to the Hasse diagram for the weak order in Figure 1.5 and in Figure 1.6. This is because the poset of regions for Coxeter arrangements is isomorphic to the weak order for Coxeter groups.

Proposition 3.4.2 *(Edelman, 1984, Corollary 4.3) Given a finite Coxeter system (W, S), the weak order on W, $(W, \leq_R)$, is isomorphic to the poset of regions $\mathrm{PR}(\mathcal{A}, B)$ on the Coxeter arrangement $\mathcal{A}_W$ for any choice of base region B.*

Although for Coxeter groups the weak order is always a lattice in the finite case, the poset of regions is not always a lattice. In fact, for some arrangements, the choice of base region determines whether or not the poset of regions is a lattice.

Example 3.4.3 Let $\mathcal{A}$ be the third arrangement in Figure 3.1. If we let the base region B be one of the "triangular" regions in $\mathscr{R}_{\mathcal{A}}$, the poset of regions $\mathrm{PR}(\mathcal{A}, B)$

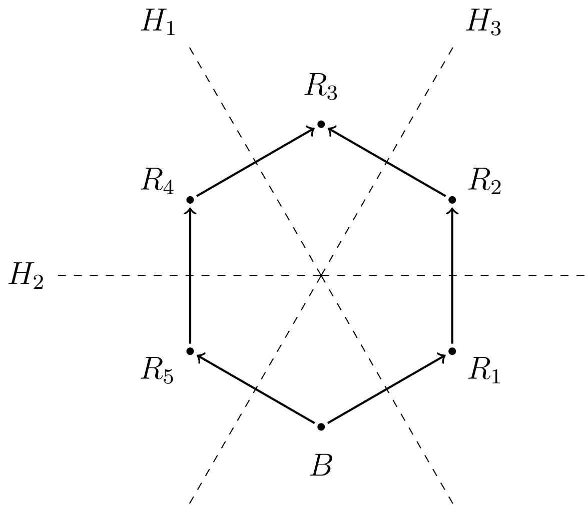


Figure 3.5: The lattice of regions associated to the type A_2 Coxeter arrangement.

is a lattice. The Hasse diagram of $\mathrm{PR}(\mathcal{A}, B)$ appears on the left in Figure 3.6. If on the other hand we let our base region B' be one of the "square" regions in $\mathscr{R}_{\mathcal{A}}$, then the poset of regions $\mathrm{PR}(\mathcal{A}, B')$ is not a lattice. The Hasse diagram of $\mathrm{PR}(\mathcal{A}, B')$ appears on the right in Figure 3.6. This poset is not a lattice since not every two elements have a join and a meet. As an example, the two circled vertices in the Hasse diagram do not have a join.

Although the poset of regions is not always a lattice, there is a large family of hyperplane arrangements whose poset of regions are lattices which we cover in the next section.

3.5 Simplicial arrangements

In this section we survey simplicial hyperplane arrangements, a family of arrangements whose poset of regions are lattices.

A *wall of a region* $R \in \mathscr{R}_{\mathcal{A}}$, or a *bounding hyperplane of* R, is a hyperplane $H \in \mathcal{A}$

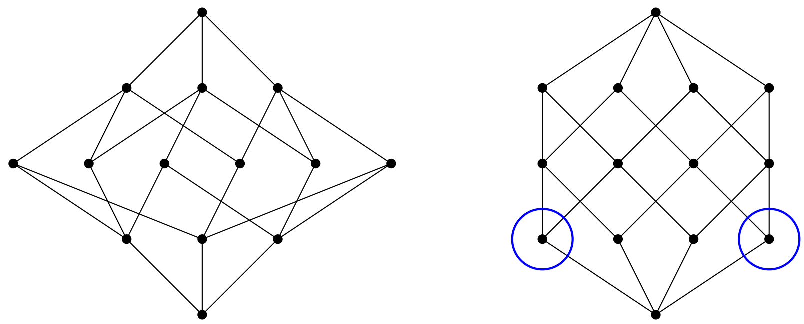


Figure 3.6: On the left is the Hasse diagram of the poset of regions of the third arrangement in Figure 3.1 where the base region B is any of the triangular regions. On the right is the Hasse diagram of the poset of regions of the same arrangement but with the base region B' as one of the square regions. The poset of regions on the left is a lattice whereas the poset of regions on the right is not since the two circled points do not have a join.

such that $\dim(H \cap R) = n - 1 = \dim(V) - 1$. Let $\mathscr{B}(R)$ denote the *set of bounding hyperplanes of a region R*. We say that a region R is *simplicial* if the set of normal vectors corresponding to its bounding hyperplanes is linearly independent. For $\mathcal{A}$ essential this is equivalent to saying that a region is simplicial if it has precisely n walls, $|\mathscr{B}(R)| = n = \dim(V)$. If every region is simplicial then we say that the arrangement is *simplicial*.

Theorem 3.5.1 *(Björner et al., 1990, Theorems 3.1 and 3.4) Suppose $\mathcal{A}$ is essential. If $\mathrm{PR}(\mathcal{A}, B)$ is a lattice then the base region B is a simplicial region. Moreover, if $\mathcal{A}$ is a simplicial arrangement then $\mathrm{PR}(\mathcal{A}, B)$ is a lattice for an arbitrary choice of base region B.*

Example 3.5.2 Considering the arrangements in Figure 3.1, the first two hyperplane arrangements are simplicial, but the third one is not. To observe this for the third hyperplane arrangement in Figure 3.1 it suffices to note that there is a region which has 4 walls directly in the middle of the figure. But since the arrangement lives in 3-space and as four vectors cannot be linearly independent in 3-space, this region is not simplicial implying the arrangement itself is not simplicial.

It turns out that Coxeter arrangements are all simplicial, see for instance (Bourbaki, 1968, Theorem VI.1.2.iii).

Theorem 3.5.3 *The Coxeter arrangement $\mathcal{A}_W$ associated to a Coxeter group W is a simplicial arrangement.*

We extend the poset of regions to the faces of an arrangement in Chapter 4. For this, we will view faces of an arrangement as covectors of an oriented matroid which we cover in the following sections.

3.6 Covectors

In this section we survey covectors: sign vectors which encode the algebraic structure of a hyperplane arrangements.

Let $\mathcal{E}$ be a (finite) ordered set and $\{-,0,+\}^{\mathcal{E}}$ be a set of (sign) vectors. The elements in $\{-,0,+\}^{\mathcal{E}}$ are called *covectors* and a subset $\mathcal{L}$ of $\{-,0,+\}^{\mathcal{E}}$ is called a *set of covectors*. For F a covector and $H \in \mathcal{E}$ let $F(H)$ denote the Hth component of F.

Example 3.6.1 As an example, let $\mathcal{E} = \{H_1, H_2\}$ be an ordered set. Then

$$\mathcal{L} = \{(+,+),(-,-),(-,+),(+,-),(+,0),(0,+),(-,0),(0,-),(0,0)\} = \{-,0,+\}^{\mathcal{E}}$$

is a set of (all) covectors. Let F be the covector $(-,0) \in \mathcal{L}$. Then $F(H_1) = -$, $F(H_2) = 0$ since the first and second components of F are $-$ and 0 respectively.

It turns out, the faces of a hyperplane arrangement can be represented as covectors. Let $\mathcal{A}$ be a (central) hyperplane arrangement and recall that e_H is a fixed normal vector for each H. Consider the *sign map* $\sigma : V \to \{-, 0, +\}^{\mathcal{A}}$ such that for $v \in V$ we have $\sigma(v) = (\sigma_H(v))_{H \in \mathcal{A}}$ where

$$\sigma_H(v) = \begin{cases} + & \text{if } \langle v, e_H \rangle > 0, \\ - & \text{if } \langle v, e_H \rangle < 0, \\ 0 & \text{if } \langle v, e_H \rangle = 0. \end{cases}$$

We extend the sign map to any face in $\mathscr{F}_{\mathcal{A}}$ by using points in the relative interior. If $\text{int}(F)$ is the set of points strictly in the interior of a face F, then the *face sign map* is the map $\hat{\sigma} : \mathscr{F}_{\mathcal{A}} \to \{-, 0, +\}^{\mathcal{A}}$ such that $\hat{\sigma}(F) = (\hat{\sigma}_H(F))_{H \in \mathcal{A}}$ where $\hat{\sigma}_H(F) = \sigma_H(x)$ for $x \in \text{int}(F)$. This map is well-defined since for arbitrary $x, y \in \text{int}(F)$ we have that $\sigma(x) = \sigma(y)$. Note that this is not the case if we were to have taken $x \in \text{int}(F)$ and y on the boundary as there could be some H such that $\sigma_H(y) = 0 \neq \sigma_H(x)$. By abuse of notation we let $\hat{\sigma}_H(F)$ be denoted by $F(H)$ as with covectors. The image of $\mathscr{F}_{\mathcal{A}}$ by the face sign map $\hat{\sigma}$ is then the set of covectors associated to the arrangement $\mathcal{A}$. We denote this set of covectors $\mathcal{L}(\mathcal{A})$.

Example 3.6.2 The set of covectors $\mathcal{L}$ in Example 3.6.1 is a set of covectors associated to a hyperplane arrangement. In particular, the hyperplane arrangement is given by $\mathcal{A} = \{H_1, H_2\} = \mathcal{E}$ and is the arrangement as in Figure 3.7. Each face of the arrangement $\mathcal{A}$ is labelled in Figure 3.7 by its associated covector in $\mathcal{L} = \mathcal{L}(\mathcal{A}) = \{-, 0, +\}^{\mathcal{A}}$. The arrangement $\mathcal{A}$ is the Coxeter arrangement of type $A_1 \times A_1$.

In the following section we survey four operations on covectors which are used heavily in Chapter 4.

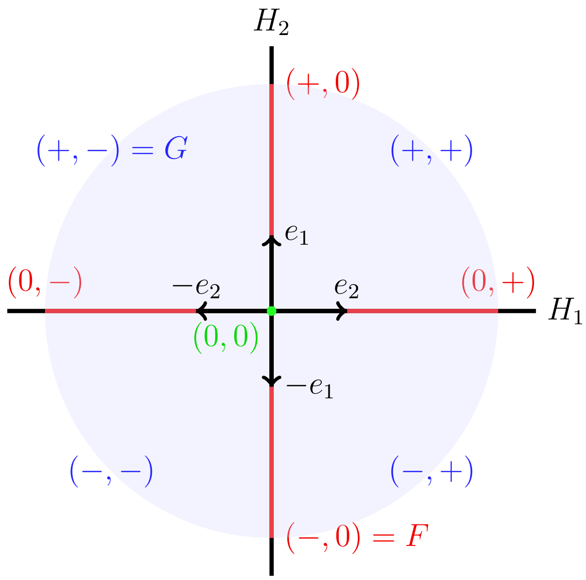


Figure 3.7: Faces of Coxeter arrangement of type $A_1 \times A_1$ labelled with covectors.

3.7 Covector Operations

In this section we survey four operations on covectors and their interpretations in terms of hyperplane arrangements. Let $\mathcal{E}$ be an ordered set and let $\mathcal{L} \subseteq \{-,0,+\}^{\mathcal{E}}$ be a set of covectors.

3.7.1 Opposite

The *opposite* of $F \in \mathcal{L}$ is the covector $-F$ such that for $H \in \mathcal{E}$

$$-F(H) = \begin{cases} + & \text{if } F(H) = -, \\ - & \text{if } F(H) = +, \\ 0 & \text{if } F(H) = 0. \end{cases}$$

In terms of a central hyperplane arrangement $\mathcal{A}$, the opposite of a face F is the face $-F \in \mathscr{F}_{\mathcal{A}}$ where $-F = \{-x \mid x \in F\}$. In other words, it is the face which is contained in every hyperplane that F is contained in and is separated from F by every other hyperplane.

Example 3.7.1 Let $\mathcal{A} = \{H_1, H_2\} = \mathcal{E}$ be the ordered set as in Example 3.6.1 associated to the Coxeter arrangement of type $A_1 \times A_1$ as in Figure 3.7. Let $F = (-, 0)$ be the covector in $\mathcal{L} = \{-, 0, +\}^{\mathcal{E}}$. The opposite of F is obtained by changing the sign of each component; $-F = (+, 0)$. From Figure 3.7 it can be observed that $-F$ is on the "opposite" side of the hyperplane arrangement. In other words, $-F$ is contained in all the same hyperplanes as F (namely H_2) and separated from F by every other hyperplane (namely H_1) as can be seen in Figure 3.7.

3.7.2 Composition

The *composition* of two covectors F and G in $\mathcal{L}$ is the covector $F \circ G$ such that for $H \in \mathcal{E}$

$$(F \circ G)(H) = \begin{cases} F(H) & \text{if } F(H) \neq 0, \\ G(H) & \text{otherwise.} \end{cases}$$

In terms of a central arrangement $\mathcal{A}$, we consider an arbitrary point p in the interior of F. The composition of F and a face G is the face in $\mathscr{F}_{\mathcal{A}}$ we land into when we move the point p slightly towards a point q in the relative interior of G.

Example 3.7.2 Continuing our example from Example 3.7.1 we have $F = (-, 0)$ and we consider another covector, say $G = (+, -) \in \mathcal{L}$. The composition of F and G is given by $F \circ G = (-, -)$. Notice that composition is not commutative since $G \circ F = (+, -) \neq (-, -) = F \circ G$. From Figure 3.7 it can be observed that $F \circ G$ is nothing more than the face we land into by starting from F and moving slightly towards G as can be seen in the following figure. This is observed in Figure 3.8 where p is a point in the interior of F and we move p slightly towards a point q in the relative interior of the face G into the face represented by the covector $(-,-)$.

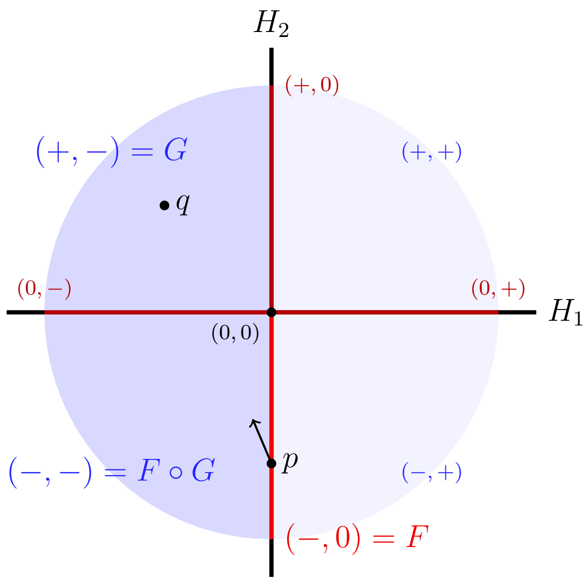


Figure 3.8: An example of composition of the face $F = (-,0)$ with the face $G = (+,-)$.

Similarly, $G \circ F = G$ is the face we land into by starting from G and moving slightly towards F.

3.7.3 Reorientation

The *reorientation* of a covector F by a covector G is the covector F_{-G} such that for $H \in \mathcal{E}$

$$(F_{-G})(H) = \begin{cases} -F(H) & \text{if } G(H) = 0, \\ F(H) & \text{otherwise.} \end{cases}$$

In terms of an arrangement $\mathcal{A}$, the reorientation of F by G is the covector of the face obtained by changing the sign of all hyperplanes which contain G. Note that reorientation by a face G which is not contained in F might give a covector which is not associated to a face in our arrangement.

Example 3.7.3 Continuing our example from Example 3.7.1, reorienting $F = (-, 0)$ by $G = (+, -)$ doesn't alter F, $F_{-G} = (-, 0) = F$, since G has no zero components. On the other hand, reorienting G by F does give us another covector since $F(H_2) = 0$, *i.e.*, $G_{-F} = (+, +)$. From Figure 3.7 it can be observed that since G is not contained in any hyperplanes then F reflects over nothing, thus $F_{-G} = F$. On the other hand, since $F \subseteq H_2$ then reorienting G by F reflects the face G over the hyperplane H_2 giving us the face associated to the covector $(+, +)$. This can be observed in Figure 3.9.

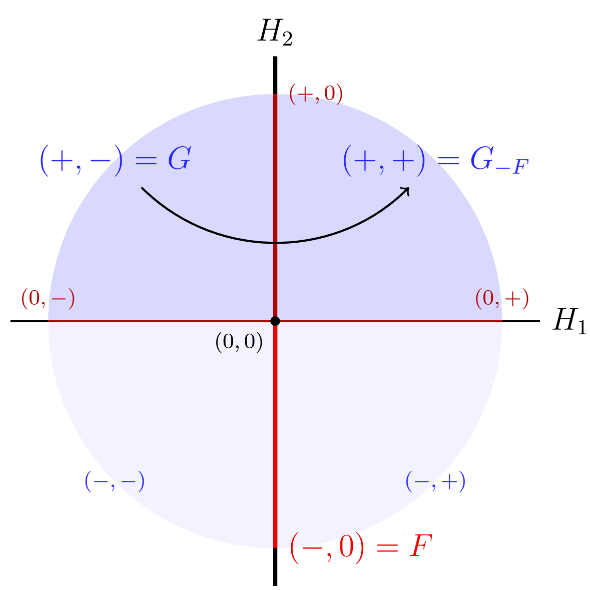


Figure 3.9: An example of reorientation of the face $F = (-, 0)$ by the face $G = (+, -)$.

3.7.4 Separation set

The *separation set* between the covectors F and G is given by the set

$$S(F,G) = \{H \in \mathcal{E} \mid \; F(H) = -G(H) \neq 0\}.$$

In terms of hyperplane arrangements, the separation set is the generalization of the separation set between regions to the set of faces. In other words, it is the set of hyperplanes which separate F and G.

Example 3.7.4 Continuing our example from Example 3.7.1, the separation set between $F = (-,0)$ and $G = (+,-)$ is given by $S(F,G) = \{H_1\}$. From Figure 3.7 it can be observed that the only hyperplane separating F and G is the hyperplane H_1.

Covectors and the covector operations we defined in this section can be used to define an oriented matroid (a generalization of central hyperplane arrangements). Oriented matroids give us an algebraic way to look at hyperplane arrangements and make calculations, such as separation set and composition, much easier. We survey the notion of an oriented matroid in the following section.

3.8 Oriented matroids

In this section we survey oriented matroids, a generalization of central hyperplane arrangements. Oriented matroids are useful as they provide an algebraic point of view for studying hyperplane arrangements.

Using covectors, the properties of (essential) central hyperplane arrangements can be generalized into oriented matroids defined as follows.

Definition 3.8.1 Let $\mathcal{E}$ be a finite ordered set and $\mathcal{L}$ a subset of $\{-,0,+\}^{\mathcal{E}}$. An *oriented matroid* is a pair $(\mathcal{E}, \mathcal{L})$ such that the set of covectors $\mathcal{L} \subseteq \{-, 0, +\}^{\mathcal{E}}$ satisfy the following properties:

(1) $\mathbf{0} \in \mathcal{L}$ where $\mathbf{0}$ is the all zero vector.

(2) If $F \in \mathcal{L}$ then $-F \in \mathcal{L}$.

(3) If F and G are in $\mathcal{L}$ then $(F \circ G) \in \mathcal{L}$.

(4) *Elimination axiom:* If F and G are in $\mathcal{L}$ and $H \in S(F, G)$ then there exists $X \in \mathcal{L}$ such that $X(H) = 0$ and $X(H') = (F \circ G)(H') = (G \circ F)(H')$ for all $H' \notin S(F, G)$.

In terms of (essential) central hyperplane arrangements, the four properties of an oriented matroid translate to the following four properties of an essential central hyperplane arrangement $\mathcal{A}$.

(1) The origin $\{0\}$ is always a face in $\mathscr{F}_{\mathcal{A}}$ since the intersection of every hyperplane is $\{0\}$ (since $\mathcal{A}$ is essential and central).

(2) The opposite of every face F is also present in the arrangement. It is the face $-F$ obtained by reflecting F over every hyperplane (as $\mathcal{A}$ is central).

(3) Similarly, from any face F we can move a point in the interior of F towards any other face G. The face this point moves to is also a face in our arrangement.

(4) Finally, the elimination axiom says that if we have two faces F and G which are separated by some hyperplane H, then there is some face X in $\mathscr{F}_{\mathcal{A}}$ which is contained in H such that X is contained in the same (closed) half-space of every hyperplane not in the separation set of F and G as F and G are. Note that this property is not trivial.

Therefore, every set of covectors $\mathcal{L}(\mathcal{A})$ associated to a central hyperplane arrangement is the set of covectors for an oriented matroid, see for instance (Björner et al., 1999, Theorem 5.1.4).

Theorem 3.8.2 *Let $\mathcal{A}$ be an (essential central) hyperplane arrangement and let $\mathcal{L}(\mathcal{A})$ be the set of covectors associated to $\mathcal{A}$. Then $(\mathcal{A}, \mathcal{L}(\mathcal{A}))$ is an oriented matroid.*

In other words, we can use the language of oriented matroids for studying hyperplane arrangements. The usefulness and efficiency of oriented matroids for hyperplane arrangements was so prominent that it took over the traditional methods of working with hyperplane arrangements. As an example, in Chapter 4 we will use the fact that, like in the case of Coxeter groups, every face of a hyperplane arrangement can be viewed as an interval in the poset of regions. Using hyperplane arrangements, this follows from (Edelman, 1984, Lemma 1.2) where the proof took half a page. On the other hand, using oriented matroids, this follows from (Björner et al., 1999, Lemma 4.2.12) where the proof was a one line direct consequence of the definition of an oriented matroid.

Oriented matroids also open the door to a larger class of objects since not every oriented matroid is associated to a hyperplane arrangement.

Example 3.8.3 An example of an oriented matroid that is not a hyperplane arrangement uses the covectors of a pseudoline arrangement. A *pseudoline* is some simple closed curve in the projective plane $\mathbb{P}^2$ whose complement is connected. A *pseudoline arrangement* is a finite set of pseudolines such that every pair of pseudolines meet at exactly one point and the intersection of all pseudolines is empty. A pseudoline arrangement is a hyperplane arrangement if every pseudoline can be "stretched" to become a straight line. In Figure 3.10 we give an example of a pseudoline arrangement that cannot be converted into a hyperplane arrangement.

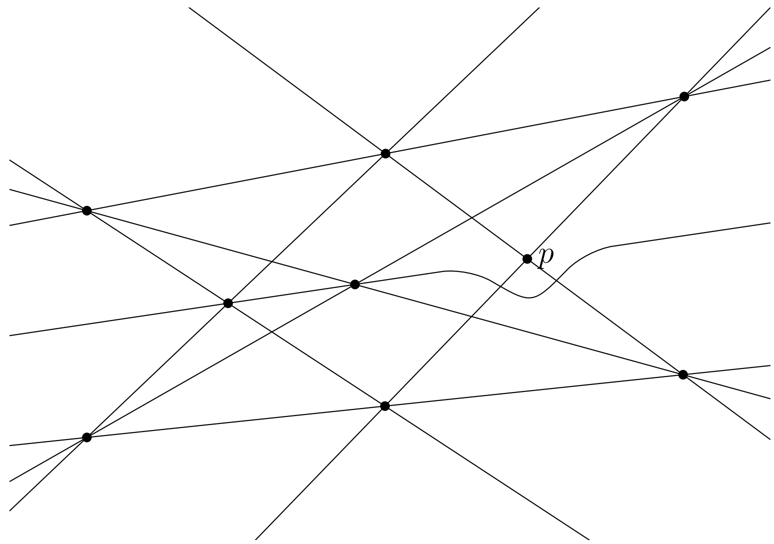


Figure 3.10: The pseudoline arrangement of an oriented matroid that does not come from a hyperplane arrangement.

This follows from the fact that we cannot straighten the horizontal curve in the middle (the only non-straight curve) without it intersecting the point p due to Pappus' hexagon theorem (see for instance (Coxeter, 1989)). In fact this is the smallest oriented matroid not associated to a hyperplane arrangement. In particular, every pseudoline arrangement with 8 or fewer pseudolines is associated to some hyperplane arrangement. Additionally, pseudoline (or in higher dimensions, pseudosphere) arrangements give a universal model for oriented matroids by the topological representation theorem (see (Björner et al., 1999, Theorem 5.2.1)). For more information on this oriented matroid, the reader is invited to consult (Björner et al., 1999, Section 1.3).

CHAPTER IV

THE FACIAL WEAK ORDER ON HYPERPLANE ARRANGEMENTS

The text in this chapter is about to be submitted and was written by myself, Christophe Hohlweg, Thomas McConville and Vincent Pilaud.

A *hyperplane arrangement* is a finite collection $\mathcal{A}$ of linear hyperplanes in a finite dimensional real vector space V. Its *regions* are the closures of the connected components of the complement in V of the union of all hyperplanes in $\mathcal{A}$. A region is *simplicial* if the normal vectors to its bounding hyperplanes are linearly independent, and the arrangement is *simplicial* if all its regions are. The *zonotope* of the arrangement $\mathcal{A}$ is a convex polytope dual to the arrangement $\mathcal{A}$, obtained as the Minkowski sum of line segments normal to the hyperplanes of $\mathcal{A}$.

The regions of a hyperplane arrangement $\mathcal{A}$ can be ordered as follows. Define the *separation set* $S(R, R')$ between two regions R and R' of $\mathcal{A}$ as the set of hyperplanes of $\mathcal{A}$ separating the two regions R and R'. For a fixed base region B, the *poset of regions* is the set of regions of $\mathcal{A}$ ordered by inclusion of their separation sets $S(B, R)$ with the base region B. A. Björner, P. H. Edelman and G. M. Ziegler (Björner et al., 1990) showed that the poset of regions is a lattice if $\mathcal{A}$ is simplicial, and that the base region B is simplicial if the poset of regions is a lattice. The Hasse diagram of the poset of regions can also be seen as the graph of the zonotope of $\mathcal{A}$, oriented from the base region B to its opposite region $-B$.

In this paper, we extend the study of the *facial weak order* $\mathbf{FW}(\mathcal{A}, B)$, as introduced in (Dermenjian et al., 2018) for Coxeter arrangements. This order is a poset structure on the faces of the hyperplane arrangement $\mathcal{A}$ or, equivalently, of the zonotope of $\mathcal{A}$. It was first introduced by D. Krob, M. Latapy, J.-C. Novelli, H.-D. Phan, and S. Schwer in (Krob et al., 2001) for the braid arrangement (the Coxeter arrangement of type A) where it was shown to be a lattice. It was then extended to arbitrary Coxeter arrangements by P. Palacios and M. Ronco in (Palacios & Ronco, 2006) and it was shown to be a lattice for arbitrary Coxeter arrangements in (Dermenjian et al., 2018). The aims of this article are to extend the facial weak order to central hyperplane arrangements.

The first part of this article, contained in § 4.1 and § 4.2, is dedicated to providing four equivalent definitions for the facial weak order on a given central hyperplane arrangement:

- in terms of *separation set* comparisons between the minimal and maximal regions incident to a face (§ 4.1.4),
- by providing a precise description of its *covering relations* (§ 4.1.5),
- in terms of *covectors* of the associated oriented matroid (§ 4.2.1),
- and in terms of *root inversion sets* of the normals to the hyperplane arrangements (§ 4.2.4).

We prove these four definitions to be equivalent in Theorem 4.2.9 and Theorem 4.2.18. In the case of a Coxeter arrangement, this recovers and expands the descriptions in (Dermenjian et al., 2018).

In § 4.3, we show that if the poset of regions of a hyperplane arrangement is a lattice, then the facial weak order is a lattice (Theorem 4.3.1). This is achieved using the BEZ lemma (Björner et al., 1990, Lemma 2.1) which states that a poset is a lattice as soon as there exists a join $x \vee y$ for every two elements x and y that both cover the same element. This extends the results of (Krob et al., 2001) for the braid arrangement and of (Dermenjian et al., 2018) for Coxeter arrangements.

For a general arrangement $\mathcal{A}$, the facial weak order may not be a lattice, but its topology still admits a nice description that we study in § 4.4. There are a wide variety of simplicial complexes associated to a hyperplane arrangement. Typically, complexes that depend on the matroid structure of $\mathcal{A}$ are homotopy equivalent to a wedge of (several) spheres, *e.g.* the independence complex, the reduced broken circuit complex, or the lattice of flats (Björner, 1992). On the other hand, complexes that depend on the *oriented* matroid structure of $\mathcal{A}$ tend to be homotopy equivalent to a single sphere or are contractible, *e.g.* the complexes of acyclic, convex, or free sets (Edelman et al., 2002), the poset of regions (Edelman, 1984), or the poset of cellular strings (Björner, 1992). We compute the homotopy types of intervals of the facial weak order (Theorem 4.4.6). Keeping with the aforementioned trends, we prove that every interval of the facial weak order is either contractible or homotopy equivalent to a sphere.

To conclude, let us mention two directions that are not explicitly explored here to keep the paper short. First, although we use the language of oriented matroids, we only deal here with facial weak order of hyperplane arrangements. The results presented here seem however to extend in the context of simple simplicial oriented matroids. Second, using the same tools as in (Dermenjian et al., 2018), one can observe that when the arrangement is simplicial, each lattice congruence of the poset of regions naturally translates to a lattice congruence of the facial weak order.

4.1 Facial weak order on the poset of regions

In this section we start by recalling classical definitions on hyperplane arrangements. For more details, we refer the reader to the book by P. Orlik and H. Terao (Orlik & Terao, 1992), the book by R. Stanley (Stanley, 2011) and the paper by A. Björner, P. H. Edelman and G. M. Ziegler (Björner et al., 1990). We then introduce the *facial weak order* and discuss its cover relations.

4.1.1 Hyperplane arrangements

Let $(V, \langle \cdot, \cdot \rangle)$ be an n-dimensional real Euclidean vector space. A *central hyperplane arrangement*, or *arrangement* for short, is a finite set $\mathcal{A}$ of linear hyperplanes in V. For each $H \in \mathcal{A}$, we choose some fixed nonzero vector e_H normal to H, that is, such that $H = \{v \in V \mid \langle e_H, v\rangle = 0\}$ (the choice of the normal vectors e_H are unique up to nonzero scalar multiplication). We also consider the two half spaces $H^+ := \{v \in V \mid \langle e_H, v\rangle > 0\}$ and $H^- := \{v \in V \mid \langle e_H, v\rangle < 0\}$ bounded by H.

The *rank* of $\mathcal{A}$ is the dimension $\operatorname{rank}(\mathcal{A})$ of the linear subspace V' spanned by the vectors e_H, for $H \in \mathcal{A}$. An arrangement $\mathcal{A}$ is *essential* if $\operatorname{rank}(\mathcal{A}) = \dim(V)$, or equivalently if the intersection of all hyperplanes of $\mathcal{A}$ is the origin. We assume our arrangements to be essential unless stated otherwise. From a combinatorial perspective the specialization to essential arrangements causes no loss of generality. This is due to the fact that for each arrangement $\mathcal{A}$ of rank m in $V \cong \mathbb{R}^n$ there is an associated essential arrangement $\mathcal{A}'$ in $V' \cong \mathbb{R}^m$ whose face structure is similar. See (Björner et al., 1999, Section 2.1) for more details.

The *regions* of an arrangement $\mathcal{A}$ are the closures of the connected components of $V \smallsetminus (\bigcup_{H \in \mathcal{A}} H)$. We denote by $\mathscr{R}_{\mathcal{A}}$ the set of regions of $\mathcal{A}$. A *wall* of a region R in $\mathcal{A}$ is a bounding hyperplane $H \in \mathcal{A}$ of R, that is, $\dim(H \cap R) = \dim(V) - 1$. A region

R is said to be *simplicial* if the normal vectors of its walls are linearly independent. If $\mathcal{A}$ is essential then a region is simplicial if and only if it has precisely $\mathrm{rank}(\mathcal{A})$ walls. An arrangement is *simplicial* if all its regions are simplicial.

A *face* of $\mathcal{A}$ is the intersection of some regions of $\mathcal{A}$. We denote by $\mathscr{F}_\mathcal{A}$ the set of faces of $\mathcal{A}$. Note that the regions are the codimension 0 faces of $\mathcal{A}$. The *face poset* of the arrangement $\mathcal{A}$ is the poset $(\mathscr{F}_\mathcal{A}, \subseteq)$ of faces of $\mathcal{A}$ ordered by inclusion. The *face lattice* of the arrangement $\mathcal{A}$ is the face poset together with the vector space itself as the maximum element. In this paper, we will consider a different poset structure on $\mathscr{F}_\mathcal{A}$.

Example 4.1.1 Well-known examples of simplicial hyperplane arrangements are the *Coxeter arrangements*. These are the hyperplane arrangements associated to a Coxeter system (W, S). See Figure 4.1 for an illustration of the Coxeter arrangements of types A_3, B_3 and H_3. We refer the reader to the books (Humphreys, 1990; Björner & Brenti, 2005) for comprehensive surveys on Coxeter groups. Figure 4.2 gives an example of the type A_2 Coxeter arrangement together with its faces. The R_i (in blue) are the six regions of the arrangement and are the codimension 0 faces. There are also six codimension 1 faces denoted by the F_i (in red) and one codimension 2 face $\{0\}$ at the centre (in green).

4.1.2 Poset of regions

Consider an arrangement $\mathcal{A}$. The *separation set* of two regions $R, R' \in \mathscr{R}_\mathcal{A}$ is

$$S(R, R') := \{H \in \mathcal{A} \mid \ H \text{ separates } R \text{ from } R'\}.$$

We now choose B to be a distinguished region of $\mathcal{A}$ called the *base region*, and abbreviate $S(B, R)$ into $S(R)$. The *poset of regions* with respect to B is the partial order $\mathrm{PR}(\mathcal{A}, B) = (\mathscr{R}_\mathcal{A}, \leq_{\mathrm{PR}})$ on the regions $\mathscr{R}_\mathcal{A}$ of the arrangement $\mathcal{A}$ defined by

$$R \leq_{\mathrm{PR}} R' \quad \iff \quad S(R) \subseteq S(R').$$

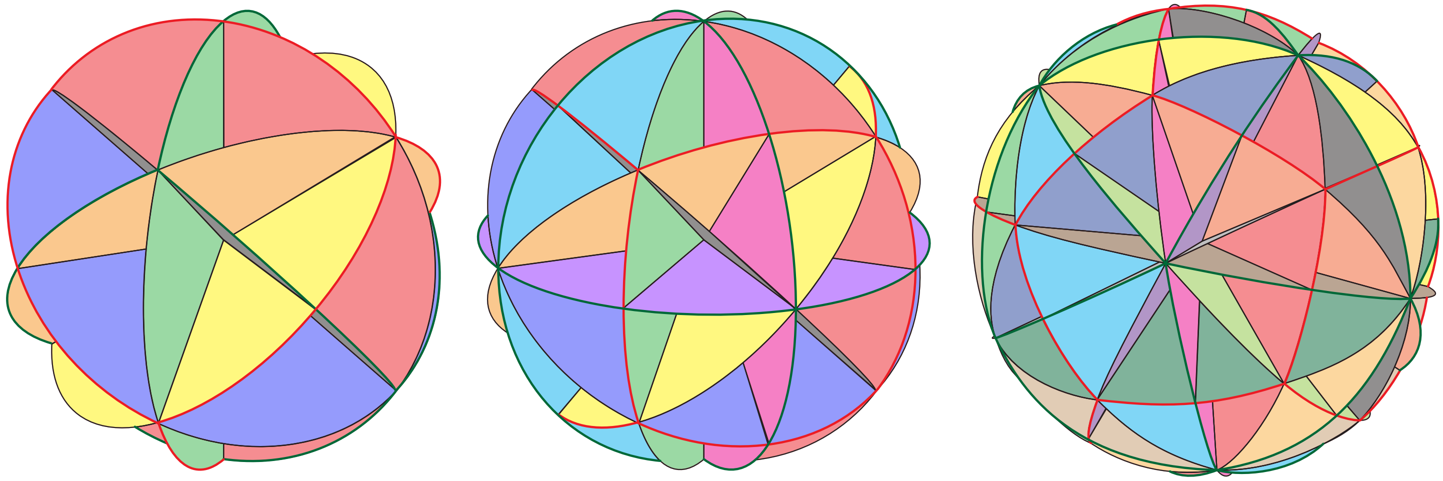


Figure 4.1: The type A_3, B_3 and H_3 Coxeter arrangements.

The poset of regions is graded by the cardinality of the separation set $|S(R)|$ of a region R. The base region B is its minimum element and has rank $|S(B)| = |\varnothing| = 0$, and its opposite region $-B$ is its maximum element and has rank $|S(-B)| = |\mathcal{A}|$. Additionally, we have the following statement, see *e.g.* (Edelman, 1984, Proposition 2.1).

Proposition 4.1.2 *The map* $R \mapsto -R := \{-v \mid v \in R\}$ *is a self-duality of the poset of regions* $\mathrm{PR}(\mathcal{A}, B)$*.*

The reader is referred to § 4.3.3 for a definition of self-dual. It is known that posets of regions associated to simplicial arrangements are lattices.

Theorem 4.1.3 ((Björner et al., 1990, Theorems 3.1 and 3.4)) *Suppose* $\mathcal{A}$ *is essential. If the poset of regions* $\mathrm{PR}(\mathcal{A}, B)$ *is a lattice then the base region* B *is a simplicial region. Moreover, if* $\mathcal{A}$ *is a simplicial arrangement then the poset of regions* $\mathrm{PR}(\mathcal{A}, B)$ *is a lattice for an arbitrary choice of base region* B*.*

Example 4.1.4 Following with Example 4.1.1, the Hasse diagram of the poset of regions of a type A_2 Coxeter arrangement is given in Figure 4.3. For Coxeter arrangements, the poset of regions is nothing more than the (right) weak order where the separation sets can be seen as inversion sets (Humphreys, 1990; Björner & Brenti, 2005). In this example, we see that $R_1 \leq_{\mathrm{PR}} R_2$ since

$$S(R_1) = \{H_1\} \subseteq \{H_1, H_2\} = S(R_2),$$

but $R_5 \not\leq_{\mathrm{PR}} R_2$ since $S(R_5) = \{H_3\} \not\subseteq \{H_1, H_2\} = S(R_2)$. The minimal element is B, the maximal element is $-B = R_3$.

4.1.3 Facial intervals

One of the interesting facts about the poset of regions is that it allows each face in $\mathscr{F}_{\mathcal{A}}$ to be described by a unique interval in $\mathrm{PR}(\mathcal{A}, B)$. These intervals will be used to define the facial weak order.

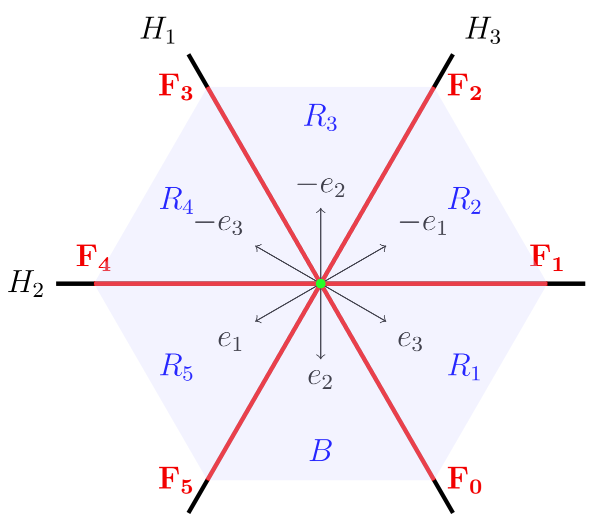


Figure 4.2: The type A_2 Coxeter arrangement where B is the intersection of the positive half-spaces of all hyperplanes. See Example 4.1.1.

Proposition 4.1.5 *For any face $F \in \mathscr{F}_{\mathcal{A}}$, the set $\{R \in \mathscr{R}_{\mathcal{A}} \mid F \subseteq R\}$ is an interval of the poset of regions* $\mathrm{PR}(\mathcal{A}, B)$. *We denote it* $[m_F, M_F]$ *and call it the facial interval of* F. *Moreover,* $F = \bigcap_{R \in [m_F, M_F]} R$.

Remark 4.1.6 A proof of the above Proposition 4.1.5 can be found in (Björner et al., 1999, Lemma 4.2.12). It is based on the following geometric idea: the region m_F (resp. M_F) is the region that is found when starting from any point in the relative interior of the face F and slightly moving in the direction of (resp. away from) a point in the relative interior of the base region B.

For instance the interval corresponding to a region is the singleton constituted of that region. Note that not all intervals of the poset of region $\mathrm{PR}(\mathcal{A}, B)$ are facial intervals; only those of the form $\{R \in \mathscr{R}_{\mathcal{A}} \mid F \subseteq R\}$ for some face $F \in \mathscr{F}_{\mathcal{A}}$. Since $F = \bigcap_{R \in [m_F, M_F]} R$, we obtain the following corollary.

Corollary 4.1.7 *For* $F, G \in \mathscr{F}_{\mathcal{A}}$, *we have* $F \subseteq G \iff [m_F, M_F] \supseteq [m_G, M_G]$.

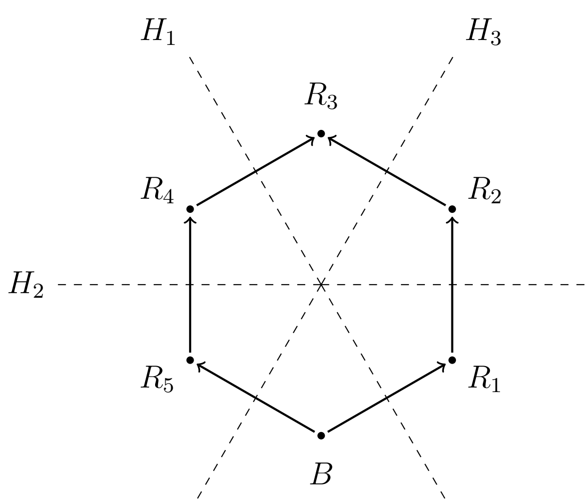


Figure 4.3: The lattice of regions associated to the type A_2 Coxeter arrangement. See Example 4.1.4.

Example 4.1.8 As we saw in Example 4.1.1, there are 13 faces in the arrangement of Figure 4.3. For instance, the origin $\{0\}$ is represented by $[B, R_3]$ and F_1 is represented with $[R_1, R_2]$. Each region R is given by the interval $[R, R]$.

4.1.4 Facial weak order

We now state the definition of the facial weak order[1] which will be the focus for the rest of the paper.

Definition 4.1.9 The *facial weak order* is the order $\leq_{\mathbf{FW}}$ on $\mathscr{F}_{\mathcal{A}}$ defined by

$$F \leq_{\mathbf{FW}} G \quad \Longleftrightarrow \quad m_F \leq_{\mathrm{PR}} m_G \quad \text{and} \quad M_F \leq_{\mathrm{PR}} M_G$$

where $[m_F, M_F]$ and $[m_G, M_G]$ are the facial intervals in $\mathrm{PR}(\mathcal{A}, B)$ associated to the faces F and G respectively. We denote by $\mathbf{FW}(\mathcal{A}, B)$ the poset $(\mathscr{F}_{\mathcal{A}}, \leq_{\mathbf{FW}})$.

Example 4.1.10 We give an example of the Hasse diagram of the facial weak order for the type A_2 Coxeter arrangement in Figure 4.4. As we saw in Example 4.1.1, there are 13 faces in the arrangement of Figure 4.3, corresponding to the 13 elements of the facial weak order. For example we have $[B, R_5] \leq_{\mathbf{FW}} [R_2, R_3]$ since $B \leq_{\mathrm{PR}} R_2$ and $R_5 \leq_{\mathrm{PR}} R_3$.

The facial weak order was first defined for the braid arrangement by D. Krob, M. Latapy, J.-C. Novelli, H.-D. Phan and S. Schwer in (Krob et al., 2001). It was then extended to arbitrary finite Coxeter arrangements by P. Palacios and M. Ronco in (Palacios & Ronco, 2006). This order was studied in de-

[1] Just like the poset of regions, it is tempting to call this order the poset of faces. However, the facial weak order IS NOT the classical face poset (the poset of faces ordered by inclusion). We have thus chosen to borrow the name facial weak order from the context of Coxeter groups studied in (Dermenjian et al., 2018) to the present context of hyperplane arrangements.

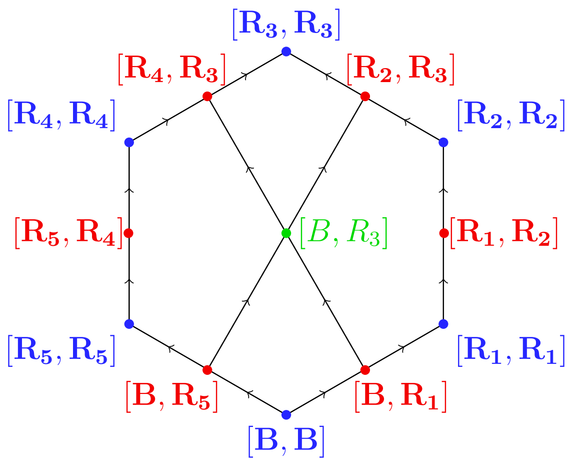


Figure 4.4: The facial weak order labelled by facial intervals for the type A_2 Coxeter arrangement. See Example 4.1.10.

tail in (Dermenjian et al., 2018). Definition 4.1.9 was written there in Coxeter language. Namely, for a Coxeter system (W, S), the poset of regions is the right weak order $\leq_R$ on elements of W. The faces of the Coxeter arrangement correspond to the standard parabolic cosets xW_I where $I \subseteq S$, $W_I = \langle I \rangle$, and $x \in W^I = \{w \in W \mid \ell(w) \leq \ell(ws)\ \forall s \in I\}$. In this case, the facial intervals are given by $[x, xw_{\circ,I}]$ where $w_{\circ,I}$ is the longest element in the parabolic subgroup W_I. The order $\leq_{\mathbf{FW}}$ was given by $xW_I \leq_{\mathbf{FW}} yW_J$ if and only if $x \leq_R y$ and $xw_{\circ,I} \leq_R yw_{\circ,J}$.

Remark 4.1.11 The facial weak order $\mathbf{FW}(\mathcal{A}, B)$ is clearly a poset (reflexive, antisymmetric and transitive) as the poset of regions is. In fact, the facial weak order $\mathbf{FW}(\mathcal{A}, B)$ is the subposet induced by facial intervals in the poset of all intervals of the poset of regions $\mathrm{PR}(\mathcal{A}, B)$ where $[a, b] < [c, d]$ if and only if $a \leq_{\mathrm{PR}} c$ and $b \leq_{\mathrm{PR}} d$.

Remark 4.1.12 Note that the poset of regions $\mathrm{PR}(\mathcal{A}, B)$ is clearly the subposet of the facial weak order $\mathbf{FW}(\mathcal{A}, B)$ induced by the singletons $[R, R]$ for $R \in \mathscr{R}_{\mathcal{A}}$.

We will see in Proposition 4.3.18 that this observation also holds at the level of lattices when $\mathcal{A}$ is simplicial.

4.1.5 Cover relations for the facial weak order

For two faces F and G such that $F \leq_{\mathbf{FW}} G$, we have $m_F \leq_{\mathrm{PR}} m_G$ and $M_F \leq_{\mathrm{PR}} M_G$ by Definition 4.1.9. The next proposition shows two types of cover relations for the facial weak order. These will be shown to be precisely all cover relations in $\mathbf{FW}(\mathcal{A}, B)$ in Theorem 4.2.9.

Proposition 4.1.13 *For any two faces $F, G \in \mathscr{F}_{\mathcal{A}}$ such that $F \leq_{\mathbf{FW}} G$, we have*

(1) $F \subseteq G \iff M_F = M_G$ and $G \subseteq F \iff m_F = m_G$,

(2) if $|\dim F - \dim G| = 1$ and $F \subseteq G$ or $G \subseteq F$, then F is covered by G.

Proof. (1) Suppose first that $F \subseteq G$. By Corollary 4.1.7, this implies the inclusion $[m_F, M_F] \supseteq [m_G, M_G]$ and therefore $M_G \leq_{\mathrm{PR}} M_F$. Furthermore, by Definition 4.1.9 since $F \leq_{\mathbf{FW}} G$ then $M_F \leq_{\mathrm{PR}} M_G$ forcing $M_F = M_G$ as desired.

Conversely, suppose $M_F = M_G$. As $F \leq_{\mathbf{FW}} G$ we have $m_F \leq_{\mathrm{PR}} m_G$ and therefore $[m_F, M_F] \supseteq [m_G, M_F] = [m_G, M_G]$. In other words $F \subseteq G$ by Corollary 4.1.7.

The equivalence $G \subseteq F \iff m_F = m_G$ is similar.

(2) Assume that $F \subseteq G$, the argument for the other case being symmetric. So $M_F = M_G$ by (1). Let $X \in \mathscr{F}_{\mathcal{A}} \smallsetminus \{F, G\}$ be a face such that $F <_{\mathbf{FW}} X <_{\mathbf{FW}} G$. By definition of the facial weak order we have $M_X = M_F = M_G$. Then $F \subseteq X \subseteq G$ by (1). Since $|\dim F - \dim G| = 1$, we necessarily have $X = F$ or $X = G$. Hence F is covered by G. □

Corollary 4.1.14 *For any two faces $F, G \in \mathscr{F}_{\mathcal{A}}$, if either $F \subseteq G$ and $M_F = M_G$*

or $G \subseteq F$ and $m_F = m_G$, then $F \leq_{\mathbf{FW}} G$. If additionally, $|\dim F - \dim G| = 1$ then F is covered by G, which we denote $F \lessdot_{\mathbf{FW}} G$.

Proof. Suppose first $F \subseteq G$ and $M_F = M_G$. For $F \leq_{\mathbf{FW}} G$ to hold, it suffices to show $m_F \leq_{\mathrm{PR}} m_G$. Since $F \subseteq G$, then by Corollary 4.1.7 $[m_F, M_F] \supseteq [m_G, M_G]$ and therefore $m_F \leq_{\mathrm{PR}} m_G$ as desired.

If $G \subseteq F$ and $m_F = m_G$, then it suffices to show $M_F \leq_{\mathrm{PR}} M_G$. Similarly, since $G \subseteq F$, then $[m_G, M_G] \supseteq [m_F, M_F]$ and therefore $M_F \leq_{\mathrm{PR}} M_G$ as desired.

Additionally, if $|\dim F - \dim G| = 1$, then by Proposition 4.1.13(2) we have $F \lessdot_{\mathbf{FW}} G$.

□

4.2 Geometric interpretations for the facial weak order

We describe in this section two different geometric interpretations for the facial weak order: first by the covectors of the corresponding oriented matroid, then by what we call root inversion sets which relates to the geometry of the corresponding zonotope. We prove along the way that these various interpretations are equivalent.

Throughout this section, $\mathcal{A}$ is a hyperplane arrangement. We fix a normal vector e_H to each hyperplane $H \in \mathcal{A}$, so that $H = \{v \in V \mid \langle e_H, v\rangle = 0\}$. We consider the half spaces $H^+ = \{v \in V \mid \langle e_H, v\rangle \geq 0\}$ and $H^- = \{v \in V \mid \langle e_H, v\rangle \leq 0\}$ where the boundary in both cases is H. For convenience, we choose the direction of the vector e_H such that the base region B lies in H^+.

4.2.1 Covectors and oriented matroids

In this section, we introduce basic oriented matroid terminology to deal geometrically with our hyperplane arrangements. As we only consider hyperplane arrangements, we focus on realizable oriented matroids. Moreover, we only consider covectors, and do not discuss other perspectives on oriented matroids. A more general setting and background on oriented matroids can be found in the book by A. Björner, M. Vergas, B. Sturmfels, N. White and G. M. Ziegler (Björner et al., 1999).

The *sign map* of the hyperplane arrangement $\mathcal{A}$ is the map

$$\sigma : V \to \{-, 0, +\}^{\mathcal{A}}$$

defined for $v \in V$ by $\sigma(v) = \Big(\sigma_H(v)\Big)_{H\in\mathcal{A}}$ where

$$\sigma_H(v) = \operatorname{sign}(\langle v, e_H\rangle) = \begin{cases} + & \text{if } \langle v, e_H\rangle > 0, \\ - & \text{if } \langle v, e_H\rangle < 0, \\ 0 & \text{if } \langle v, e_H\rangle = 0. \end{cases}$$

This map may be extended to assign to each face of $\mathcal{A}$ a vector in $\{-, 0, +\}^{\mathcal{A}}$ as follows. Denote by $\operatorname{int}(F)$ the set of points in the relative interior of the face F. The *face sign map* of the hyperplane arrangement $\mathcal{A}$ is the map

$$\hat{\sigma} : \mathscr{F}_{\mathcal{A}} \to \{-, 0, +\}^{\mathcal{A}}$$

defined by $\hat{\sigma}(F) = \sigma(v)$ for $v \in \operatorname{int}(F)$. This map is well-defined since, for arbitrary v and w in $\operatorname{int}(F)$, we have that $\sigma(v) = \sigma(w)$. However, note that for v on the boundary of F we could have that $\sigma_H(v) = 0$ even if $\hat{\sigma}_H(F) \neq 0$ for $H \in \mathcal{A}$. Thus for $v \in F \smallsetminus \operatorname{int}(F)$ either $\sigma_H(v) = \hat{\sigma}_H(F)$ or $\sigma_H(v) = 0$ for each $H \in \mathcal{A}$. Note that a face F is easily recovered from its covector:

$$F = \bigcap_{H\in\mathcal{A}} H^{\hat{\sigma}_H(F)}.$$

By abuse of notation we let $\hat{\sigma}_H(F)$ be denoted by $F(H)$. The sign vector $\hat{\sigma}(F)$ is called *covector* of the face F, and the image $\mathcal{L}(\mathcal{A}) := \hat{\sigma}(\mathscr{F}_{\mathcal{A}})$ of all faces in $\mathscr{F}_{\mathcal{A}}$ by the face sign map $\hat{\sigma}$ is the *set of covectors* of the hyperplane arrangement $\mathcal{A}$.

Example 4.2.1 We have represented in Figure 4.5 the covectors of all faces of the type A_2 Coxeter arrangement in Figure 4.2.

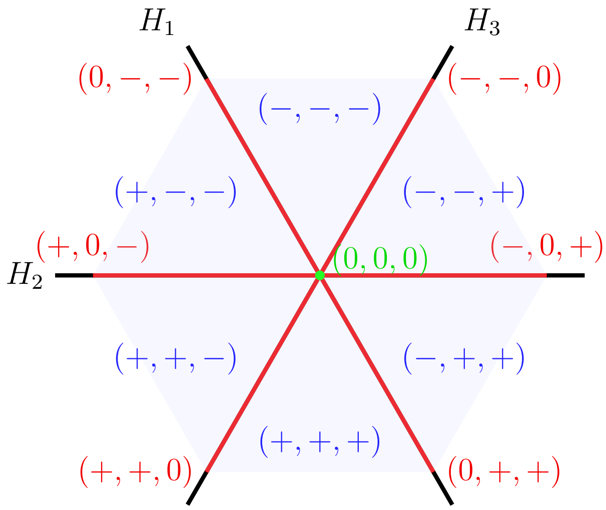


Figure 4.5: The type A_2 Coxeter arrangement where the faces are identified by their associated covectors. See Example 4.2.1.

We next define some useful operations on sign vectors which we use throughout this paper. For two sign vectors $F, G \in \{-, 0, +\}^{\mathcal{A}}$, define

- the *opposite* of F: $-F(H) = \begin{cases} + & \text{if } F(H) = -, \\ - & \text{if } F(H) = +, \\ 0 & \text{if } F(H) = 0. \end{cases}$
- the *composition* of F and G: $(F \circ G)(H) = \begin{cases} F(H) & \text{if } F(H) \neq 0, \\ G(H) & \text{otherwise.} \end{cases}$

- the *reorientation* of a F by G: $(F_{-G})(H) = \begin{cases} -F(H) & \text{if } G(H) = 0, \\ F(H) & \text{otherwise.} \end{cases}$
- the *separation set*: $S(F,G) = \{H \in \mathcal{A} \mid \; F(H) = -G(H) \neq 0\}$.

Example 4.2.2 For instance, on the arrangement of Figure 4.5, for $F = (-,0,+)$ and $G = (0,-,-)$, we have

$$-F = (+,0,-), \quad F \circ G = (-,-,+), \quad F_{-G} = (+,0,+) \quad \text{and} \quad S(F,G) = \{H_3\}.$$

Note that if F and G are covectors in $\mathcal{L}(\mathcal{A})$, then the opposite $-F$ of F and the composition $F \circ G$ of F and G are both covectors in $\mathcal{L}(\mathcal{A})$, or in other words faces in $\mathscr{F}_{\mathcal{A}}$. If furthermore, $G \subseteq F$ then the reorientation F_{-G} of F by G is a covector in $\mathcal{L}(\mathcal{A})$ as well. Moreover, $G \subseteq F_{-G}$ as faces. Note that the separation set of regions from § 4.1.2 is the same separation set as given for covectors regions. It is well-known that the set of covectors $\mathcal{L}(\mathcal{A})$ of the arrangement $\mathcal{A}$ is an oriented matroid in the sense of the following definition. Cryptomorphic definitions for oriented matroids can be found in the book by A. Björner, M. Vergnas, B. Sturmfels, N. White and G. M. Ziegler (Björner et al., 1999).

Definition 4.2.3 An *oriented matroid* is a pair $\left(\mathcal{A}, \mathcal{L}\right)$ where $\mathcal{L}$ is a collection of sign vectors in $\{-,0,+\}^{\mathcal{A}}$ satisfying the following four properties:

(1) $\mathbf{0} \in \mathcal{L}$.

(2) If $F \in \mathcal{L}$ then $(-F) \in \mathcal{L}$.

(3) If $F, G \in \mathcal{L}$ then $(F \circ G) \in \mathcal{L}$.

(4) *Elimination axiom:* If $F, G \in \mathcal{L}$ and $H \in S(F,G)$ then there exists $X \in \mathcal{L}$ such that $X(H) = 0$ and $X(H') = (F \circ G)(H') = (G \circ F)(H')$ for all $H' \notin S(F,G)$.

The notion of oriented matroids allows us to have a nice algebraic interpretation of what it means for a face F to be a face of G. We can either do a comparison between the two faces relative to the hyperplanes or we check how their covectors interact through composition.

Proposition 4.2.4 *The following assertions are equivalent for two faces $F, G \in \mathscr{F}_{\mathcal{A}}$:*

(1) $F \subseteq G$ as faces,

(2) for all $H \in \mathcal{A}$ either $F(H) = 0$ or $F(H) = G(H)$, and

(3) $G = F \circ G$ as covectors.

Proof. The equivalence of (3) and (2) is readily seen by definition of the composition $F \circ G$. Furthermore, if $F \subseteq G$ as faces then it is readily seen that for all $H \in \mathcal{A}$ either $F(H) = 0$ or $F(H) = G(H)$.

It remains to show that (2) implies (1). Suppose contrarily that $F \not\subseteq G$. If $F \supsetneq G$ then there exists some $H \in \mathcal{A}$ such that $G(H) = 0 \neq F(H)$ since $F \neq G$. Else if $F \not\supseteq G$ and $F \not\subseteq G$ there is an $H' \in \mathcal{A}$ which separates G and F. In other words $0 \neq G(H') = -F(H')$. □

In fact, the sign of a face relative to a hyperplane tells us a lot about the regions containing the face. Recall that for a region R, the separation set between the base region B and R is denoted by $S(R)$.

Lemma 4.2.5 *For a face $F \in \mathscr{F}_{\mathcal{A}}$ with facial interval $[m_F, M_F]$ and a hyperplane $H \in \mathcal{A}$,*

(1) $F(H) = -$ if and only if $H \in S(m_F)$,

(2) $F(H) = 0$ if and only if $H \in S(M_F)$ and $H \notin S(m_F)$,

(3) $F(H) = +$ *if and only if* $H \notin S(M_F)$.

In other words,

$$S(m_F) = \{H \in \mathcal{A} \mid \; F(H) < 0\} \quad \textit{and} \quad S(M_F) = \{H \in \mathcal{A} \mid \; F(H) \leq 0\}\,.$$

Proof. We show the first case, the other two being similar. First, recall that by the definition of interval, $H \in S(m_F)$ if and only if for all $R \in [m_F, M_F]$ then $H \in S(R)$. In other words, if and only if H separates the base region B from F. This is true if and only if for some $v \in \text{int}(F)$ then $\langle v, e_H \rangle < 0$ since $B \subseteq H^+$ by our chosen orientation given at the beginning of this section. In other words, if and only if $F(H) = -$. □

This lemma allows us to be a little more precise as to which faces are faces of B and gives us a stronger method of finding these faces. Not only are the faces of B the faces with all non-negative components in their covector, but, in fact, we can strengthen this by only needing to look at the hyperplanes which bound B.

Corollary 4.2.6 *The following assertions are equivalent for a face* $F \in \mathscr{F}_{\mathcal{A}}$*:*

(1) $F \subseteq B$,

(2) $F(H) \geq 0$ *for all* H *bounding* B*, and*

(3) $F(H) \geq 0$ *for all* $H \in \mathcal{A}$.

Proof. The points (1) and (3) are equivalent by Proposition 4.2.4 and the fact that $B(H) > 0$ for all $H \in \mathcal{A}$. Additionally, (3) implies (2) is readily seen.

To show that (2) implies (3), let $\mathscr{B}$ be the set of boundary hyperplanes of B. Suppose that there exists $H \in \mathcal{A} \smallsetminus \mathscr{B}$ such that $F(H) = -$. Then $H \in S(m_F)$ by Lemma 4.2.5, therefore $m_F \neq B$. This implies $H' \in S(m_F)$ for some $H' \in \mathscr{B}$

as well, since some $H' \in \mathscr{B}$ must separate B and m_F by definition of $\mathscr{B}$. In other words, $F(H') = -$ for some $H' \in \mathscr{B}$. $\square$

We conclude with an observation which ensures that a face is not contained in a given hyperplane.

Lemma 4.2.7 *Let F and G be two distinct faces in $\mathscr{F}_{\mathcal{A}}$. If there exists $H \in \mathcal{A}$ such that $G = F \cap H$, then $F(H) \neq 0$.*

Proof. Suppose contrarily that $F(H) = 0$. Then $F = F \cap H = G$ contradicting that F and G are distinct. $\square$

4.2.2 Covectors and the facial weak order

It is well-known that the face poset of the arrangement $\mathcal{A}$ can be interpreted as the poset of covectors of $\mathcal{L}(\mathcal{A})$ ordered coordinatewise by $0 < -$ and $0 < +$. Adding a maximum element to both posets allows us to interpret the face lattice as a lattice of covectors. Here, we consider instead a twisted order that relates to the facial weak order.

Definition 4.2.8 Given two covectors $F, G \in \mathcal{L}(\mathcal{A})$, let the order $\leq_{\mathcal{L}}$ be defined by

$$F \leq_{\mathcal{L}} G \quad \iff \quad G(H) \leq F(H) \text{ for all } H \in \mathcal{A},$$

where the order on signs is the natural order $- < 0 < +$.

We are ready to state our first main theorem, stating the equivalence between three descriptions of the facial weak order using Definition 4.1.9, Corollary 4.1.14 and Definition 4.2.8 respectively.

Theorem 4.2.9 *The following assertions are equivalent for two faces $F, G \in \mathscr{F}_{\mathcal{A}}$:*

(1) $F \leq_{\mathbf{FW}} G$ *in the facial weak order* $\mathbf{FW}(\mathcal{A}, B)$*,*

(2) there exists a sequence of faces $F = F_1, F_2, \ldots, F_n = G$ *such that for each* i*,* $|\dim F_i - \dim F_{i+1}| = 1$ *and either* $F_i \subseteq F_{i+1}$ *and* $M_{F_i} = M_{F_{i+1}}$ *or* $F_{i+1} \subseteq F_i$ *and* $m_{F_i} = m_{F_{i+1}}$*.*

(3) $F \leq_{\mathcal{L}} G$ *in terms of covectors.*

Due to this theorem, the two classes of cover relations from Corollary 4.1.14 describe all the cover relations for the facial weak order.

Corollary 4.2.10 *For two faces* $F, G \in \mathscr{F}_{\mathcal{A}}$*, we have* $F \lessdot_{\mathbf{FW}} G$ *in the facial weak order if and only if* $|\dim F - \dim G| = 1$ *and either* $F \subseteq G$ *and* $M_F = M_G$ *or* $G \subseteq F$ *and* $m_F = m_G$ *if and only if* $F \leq_{\mathbf{FW}} G$*,* $|\dim F - \dim G| = 1$ *and either* $F \subseteq G$ *or* $G \subseteq F$*.*

Before proving Theorem 4.2.9, we need the following two lemmas.

Lemma 4.2.11 *For* $F, G \in \mathcal{L}(\mathcal{A})$*, if* $F \leq_{\mathcal{L}} G$ *then* $F \leq_{\mathcal{L}} F \circ G \leq_{\mathcal{L}} G \circ F \leq_{\mathcal{L}} G$*.*

Proof. Suppose $F \leq_{\mathcal{L}} G$, *i.e.*, for all $H \in \mathcal{A}$, we have $G(H) \leq F(H)$. Then

- if $G(H) = +$ then $G(H) = (G \circ F)(H) = (F \circ G)(H) = F(H) = +$,
- if $G(H) = 0$ then $G(H) \leq (G \circ F)(H) = (F \circ G)(H) = F(H)$,
- if $G(H) = -$ then $G(H) = (G \circ F)(H) = - \leq (F \circ G)(H) \leq F(H)$. The first inequality is an equality when $F(H) = 0$ else the second inequality becomes an equality.

Therefore in all three cases we have $G(H) \leq (G \circ F)(H) \leq (F \circ G)(H) \leq F(H)$ for arbitrary H giving us the desired result. $\square$

Lemma 4.2.12 *If $F <_{\mathcal{L}} G$ and $S(F, G) \neq \varnothing$, then there exists $F <_{\mathcal{L}} X <_{\mathcal{L}} G$.*

Proof. Since $S(F, G)$ is non-empty, by the elimination axiom in Definition 4.2.3 for each $H \in S(F, G)$ there exists a $X \in \mathcal{L}(\mathcal{A})$ such that $X(H) = 0$ and for all $H' \notin S(F, G)$ then $X(H') = (F \circ G)(H') = (G \circ F)(H')$. Thus let H be an arbitrary hyperplane in $S(F, G)$ and let X be the associated covector in $\mathcal{L}(\mathcal{A})$.

Since $H \in S(F, G)$ and $G(H) \leq F(H)$ we are forced to have $G(H) = -$ and $F(H) = +$. Furthermore, since $X(H) = 0$ we see that our three faces are distinct, $F \neq X \neq G$. It therefore suffices to show that $G(H') \leq X(H') \leq F(H')$ for all $H' \in \mathcal{A} \setminus \{H\}$.

Suppose first that $H' \in S(F, G)$. Since $G(H') \leq F(H')$ and $F(H') = -G(H') \neq 0$ then $G(H') = -$ and $F(H') = +$. Thus $G(H') \leq X(H') \leq F(H')$ as desired.

Suppose next that $H' \notin S(F, G)$. Since $G(H') \leq F(H')$ and $-G(H') \neq F(H')$ or $G(H') = F(H') = 0$, there are three cases to consider:

- if $F(H') = G(H')$ then $F(H') = G(H') = (F \circ G)(H') = X(H')$,
- if $F(H') = 0$ and $G(H') = -$ then $X(H') = (F \circ G)(H') = G(H') < F(H)$,
- if $F(H') = +$ and $G(H') = 0$ then $X(H') = (F \circ G)(H') = F(H') > G(H)$.

Therefore $G(H') \leq X(H') \leq F(H')$ and thus $F <_{\mathcal{L}} X <_{\mathcal{L}} G$. $\square$

We now prove Theorem 4.2.9.

Proof of Theorem 4.2.9. We show that the points (1), (2), and (3) are equivalent by showing the implications (2) $\Rightarrow$ (1) $\Rightarrow$ (3) $\Rightarrow$ (2).

(2) ⇒ (1) By Corollary 4.1.14 the sequence $F_1, \ldots, F_n$ gives a a chain of covers $F = F_1 \lessdot_{\mathbf{FW}} F_2 \lessdot_{\mathbf{FW}} \cdots \lessdot_{\mathbf{FW}} F_n = G$ and therefore $F \leq_{\mathbf{FW}} G$ as desired.

(1) ⇒ (3) Suppose $F \leq_{\mathbf{FW}} G$ in the facial weak order, *i.e.*, $m_F \leq_{\mathrm{PR}} m_G$ and $M_F \leq_{\mathrm{PR}} M_G$. To show $F \leq_{\mathcal{L}} G$ it suffices to show $G(H) \leq F(H)$ for arbitrary hyperplane $H \in \mathcal{A}$. If $G(H) = -$, then $G(H) \leq F(H)$ always. If $G(H) = +$ then by Lemma 4.2.5, $H \notin S(M_G)$. But since $M_F \leq_{\mathrm{PR}} M_G$ then $S(M_F) \subseteq S(M_G)$, in other words, $H \notin S(M_F)$. Applying Lemma 4.2.5 again gives $F(H) = +$. Finally, if $G(H) = 0$ then by Lemma 4.2.5, $H \in S(M_G) \smallsetminus S(m_G)$. Therefore $H \notin S(m_G)$ and since $m_F \leq_{\mathrm{PR}} m_G$ we get $H \notin S(m_F)$. Thus by Lemma 4.2.5, $F(H) \neq -$ and $G(H) = 0 \leq F(H)$ as desired.

(3) ⇒ (2) We do this by induction on the path length from F to G. Our base case of $F = G$ trivially holds. Suppose now that $F <_{\mathcal{L}} G$. By Lemma 4.2.11, we have $F \leq_{\mathcal{L}} F \circ G \leq_{\mathcal{L}} G$. There are three cases two consider:

- Suppose first that our inequalities are strict, *i.e.*, $F <_{\mathcal{L}} F \circ G <_{\mathcal{L}} G$. Then by induction $F <_{\mathcal{L}} F \circ G$ and $F \circ G <_{\mathcal{L}} G$ gives a chain of covers $\lessdot_{\mathbf{FW}}$ such that $F = F_1 \lessdot_{\mathbf{FW}} \cdots \lessdot_{\mathbf{FW}} F_i \lessdot_{\mathbf{FW}} F \circ G \lessdot_{\mathbf{FW}} G_1 \lessdot_{\mathbf{FW}} \cdots \lessdot_{\mathbf{FW}} G_j = G$.

- If $G = F \circ G$, then by Proposition 4.2.4, $F \subseteq G$. In particular, there exists a chain of faces, $F = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = G$ such that $|\dim F_i - \dim F_{i-1}| = 1$ for all i by the face lattice being graded. It remains to show $M_{F_i} = M_{F_{i+1}}$. Since $F_i \subseteq F_{i+1}$ then for each $H \in \mathcal{A}$, either $F_i(H) = 0$ or $F_i(H) = F_{i+1}(H)$. If $F_i(H) = F_{i+1}(H)$ then $H \in S(M_F)$ if and only if $H \in S(M_{F_{i+1}})$. If $F_i(H) = 0$ then $F_{i+1}(H) = -$ (else $F(H) = 0$ and $G(H) = +$ by inclusion, contradicting the fact that $F <_{\mathcal{L}} G$). By Lemma 4.2.5 $F_i(H) = 0$ implies $H \in S(M_{F_i})$ and $F_{i+1}(H) = -$ implies $H \in S(M_{F_{i+1}})$. Therefore $S(M_{F_i}) = S(M_{F_{i+1}})$ implying $M_{F_i} = M_{F_{i+1}}$ as desired.

- If $F = F \circ G$ we have two further cases to consider. First, if
$$S(F,G) = \{H \in \mathcal{A} \mid \ F(H) = -G(H) \neq 0\} = \varnothing$$
then $G \circ F = F \circ G = F$. In particular, by Proposition 4.2.4, as $F = G \circ F$ then $G \subseteq F$ and using the sequence of faces $G = F_n, F_{n-1}, \ldots, F_1 = F$, then as in the previous case we have $|\dim F_i - \dim F_{i+1}|$, $F_i \supseteq F_{i+1}$ and $m_{F_i} = m_{F_{i+1}}$ as desired. Finally, suppose $S(F,G) \neq \varnothing$. By Lemma 4.2.12 there exists a X such that $F <_{\mathcal{L}} X <_{\mathcal{L}} G$. Thus, by inducting on this gives us the desired result. □

Moreover, using the covector definition, we show that the structure of an interval in $\mathbf{FW}(\mathcal{A}, B)$ is not altered by a change of base region as long as the new region is below the bottom element of our interval.

Proposition 4.2.13 *Let X, Y be covectors in $\mathcal{L}(\mathcal{A})$ such that $X \leq_{\mathbf{FW}} Y$ in $\mathbf{FW}(\mathcal{A}, B)$. If B' is a region such that $B' \leq_{\mathbf{FW}} X$ in $\mathbf{FW}(\mathcal{A}, B)$, then the intervals $[X, Y]$ in $\mathbf{FW}(\mathcal{A}, B)$ and in $\mathbf{FW}(\mathcal{A}, B')$ are isomorphic.*

Proof. Changing the base region from B to B' switches the orientation on any hyperplane in the separation set $S(B, B')$ and leaves the other hyperplanes with the same orientation. Since $B' \leq_{\mathbf{FW}} X$, we have $X(H) = -$ whenever $H \in S(B, B')$ where B is the base region. Hence, $Z(H) = -$ as well whenever $X \leq_{\mathbf{FW}} Z$. After the reorientation, $X(H) = + = Z(H)$ for $H \in S(B, B')$. As the orientations of the hyperplanes not in $S(B, B')$ are unchanged, we conclude that the interval $[X, Y]$ is the same in $\mathbf{FW}(\mathcal{A}, B')$ as in $\mathbf{FW}(\mathcal{A}, B)$. □

We finally derive a criterion to compare two faces of the base region B in the facial weak order.

Corollary 4.2.14 *For any faces F, G of the base region B, we have $F \supseteq G$ if and only if $F \leq_{\mathbf{FW}} G$. Similarly, for any faces F, G of the region $-B$ opposite to the base region B, we have $F \subseteq G$ if and only if $F \leq_{\mathbf{FW}} G$.*

Proof. Consider a hyperplane $H \in \mathcal{A}$. Since F is a face of the base region B, we have $F(H) \geq 0$ by Corollary 4.2.6. Since $F \supseteq G$, we have $G(H) = 0$ or $G(H) = F(H)$ by Proposition 4.2.4. Therefore, $F(H) \geq G(H)$ in both cases. We conclude that $F \leq_{\mathbf{FW}} G$. The converse can be deduced from Proposition 4.2.4. The proof for the second assertion is identical. □

4.2.3 Root inversion sets

We now provide an alternative combinatorial encoding of the covectors in terms of certain sets of normal vectors that will be related to the geometry of the corresponding zonotope in the next section. Recall that, by convention in this paper, e_H is the fixed normal vector to the hyperplane $H \in \mathcal{A}$ such that the base region B lies in H^+. We need the following three sets:

$$\Phi^+_{\mathcal{A}} := \{e_H \mid H \in \mathcal{A}\}, \quad \Phi^-_{\mathcal{A}} := \{-e_H \mid H \in \mathcal{A}\}, \quad \text{and} \quad \Phi_{\mathcal{A}} := \Phi^+_{\mathcal{A}} \cup \Phi^-_{\mathcal{A}}.$$

We call the elements in $\Phi_{\mathcal{A}}$ the *roots*[2] of the arrangement $\mathcal{A}$ and the elements in $\Phi^+_{\mathcal{A}}$ and $\Phi^-_{\mathcal{A}}$ the positive and negative roots respectively. For $X \subseteq \Phi_{\mathcal{A}}$, we denote by $X^+ := X \cap \Phi^+_{\mathcal{A}}$ the positive part and by $X^- := X \cap \Phi^-_{\mathcal{A}}$ the negative part. An example of this construction is given in Figure 4.2 where the roots give the root system for the type A_2 Coxeter arrangement.

Definition 4.2.15 The *root inversion set* of a face $F \in \mathscr{F}_{\mathcal{A}}$ is

$$\mathbf{R}(F) = \{e \in \Phi_{\mathcal{A}} \mid \langle x, e \rangle \leq 0, \text{ for some } x \in \operatorname{int}(F)\}.$$

[2] This terminology is once again inherited from Coxeter systems, but it should be noted that these roots do not necessarily form root systems.

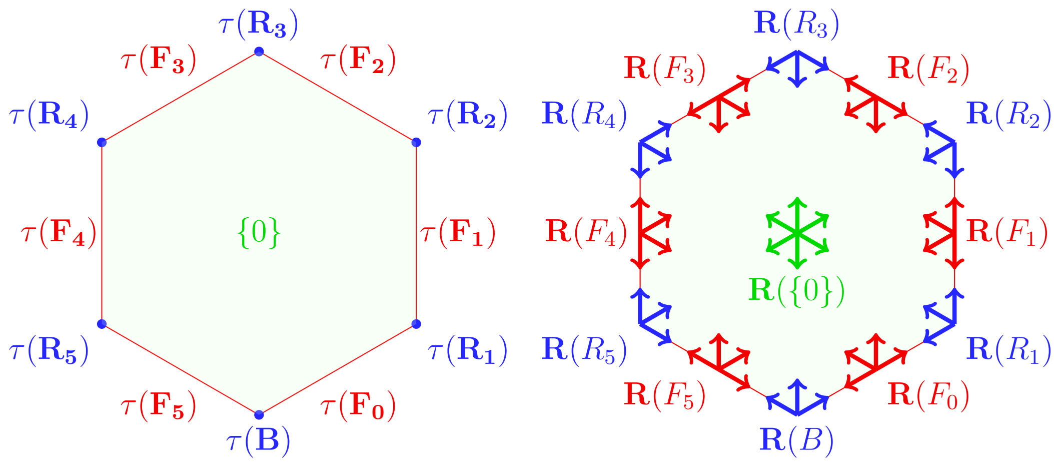


Figure 4.6: The type A_2 Coxeter arrangement. On the left is the zonotope created by the τ map in Lemma 4.2.21. On the right we label each face with the root inversion set for that face. See Example 4.2.20 and Example 4.2.23.

The following lemma shows the relationship between a root being present in a root inversion set and the sign of the covector for the associated hyperplane. An example of this relationship can be seen in Figure 4.5 and Figure 4.6.

Lemma 4.2.16 *For any $F \in \mathscr{F}_{\mathcal{A}}$ and $H \in \mathcal{A}$,*

(1) $F(H) = -$ if and only if $e_H \in \mathbf{R}(F)$ and $-e_H \notin \mathbf{R}(F)$.

(2) $F(H) = 0$ if and only if $e_H \in \mathbf{R}(F)$ and $-e_H \in \mathbf{R}(F)$.

(3) $F(H) = +$ if and only if $e_H \notin \mathbf{R}(F)$ and $-e_H \in \mathbf{R}(F)$.

In other words,

$$\mathbf{R}(F)^+ = \{e_H \mid \ H \in \mathcal{A},\ F(H) \leq 0\} \quad \text{and} \quad \mathbf{R}(F)^- = \{-e_H \mid \ H \in \mathcal{A},\ F(H) \geq 0\}.$$

Proof. We show the first case, the other cases being similar. Recall that $e \in \mathbf{R}(F)$ if and only if $\langle x, e\rangle \leq 0$ for $x \in \operatorname{int}(F)$. Furthermore, since $-e \notin \mathbf{R}(F)$ we have $\langle x, e\rangle < 0$. By definition of the sign map, since $\langle x, e\rangle < 0$ we have $\sigma_H(x) = -$, *i.e.*, $F(H) = -$ as desired.

Conversely if $F(H) = -$ then $\sigma_H(x) = -$ for $x \in \mathrm{int}(F) \subseteq F$. Then $\langle x, e\rangle < 0$ implying that $e \in \mathbf{R}(F)$. Furthermore, $\langle x, -e\rangle > 0$ gives $-e \notin \mathbf{R}(F)$ as desired. □

Corollary 4.2.17 *For any $F \in \mathscr{F}_{\mathcal{A}}$ and $e \in \Phi_{\mathcal{A}}$, we have $\mathbf{R}(F) \cap \{e, -e\} \neq \varnothing$.*

Following up Theorem 4.2.9, we are now ready to show our second main result, providing two more equivalent descriptions of the facial weak order. Recall that for $X \subseteq \Phi_{\mathcal{A}}$, we set $X^+ := X \cap \Phi^+_{\mathcal{A}}$ and $X^- := X \cap \Phi^-_{\mathcal{A}}$.

Theorem 4.2.18 *The following assertions are equivalent for two faces $F, G \in \mathscr{F}_{\mathcal{A}}$:*

(3) $F \leq_{\mathcal{L}} G$ in terms of covectors,

(4) $\mathbf{R}(F) \smallsetminus \mathbf{R}(G) \subseteq \Phi^-_{\mathcal{A}}$ and $\mathbf{R}(G) \smallsetminus \mathbf{R}(F) \subseteq \Phi^+_{\mathcal{A}}$,

(5) $\mathbf{R}(F)^+ \subseteq \mathbf{R}(G)^+$ and $\mathbf{R}(F)^- \supseteq \mathbf{R}(G)^-$.

Proof of Theorem 4.2.18. The points (4) and (5) are clearly equivalent. We thus just need to prove the equivalence between (3) and (5).

(3) ⇒ (5) Assume that $F \leq_{\mathcal{L}} G$ so that $G(H) \leq F(H)$ for all $H \in \mathcal{A}$ by Theorem 4.2.9. Then for any $H \in \mathcal{A}$, we obtain by Lemma 4.2.16 that

- if $e_H \in \mathbf{R}(F)$, then $F(H) \leq 0$, so that $G(H) \leq 0$, so that $e_H \in \mathbf{R}(G)$,
- if $-e_H \in \mathbf{R}(G)$, then $G(H) \geq 0$, so that $F(H) \geq 0$, so that $-e_H \in \mathbf{R}(F)$.

Therefore $\mathbf{R}(F)^+ \subseteq \mathbf{R}(G)^+$ and $\mathbf{R}(F)^- \supseteq \mathbf{R}(G)^-$.

(5) ⇒ (3) Assume that $\mathbf{R}(F)^+ \subseteq \mathbf{R}(G)^+$ and $\mathbf{R}(F)^- \supseteq \mathbf{R}(G)^-$. Then for any $H \in \mathcal{A}$, we obtain by Lemma 4.2.16 that

- if $G(H) = +$, then $e_H \notin \mathbf{R}(G)$, so that $e_H \notin \mathbf{R}(F)$, so that $F(H) = +$,
- if $G(H) = 0$, then $-e_H \notin \mathbf{R}(G)$, so that $-e_H \notin \mathbf{R}(F)$, so that $F(H) \geq 0$.

Therefore $G(H) \leq F(H)$ for all $H \in \mathcal{A}$, so that $F \leq_{\mathcal{L}} G$ as desired. □

4.2.4 Zonotopes

We conclude this section with an interpretation of the root inversion sets in terms of the geometry of certain polytopes associated to hyperplane arrangements.

Recall that a *polytope* is the convex hull of finitely many points in V, or equivalently a bounded intersection of finitely many half-spaces of V. The faces of P are its intersections with its supporting hyperplanes (and the faces $\varnothing$ and P itself), and its facets are its codimension 1 faces. For a face F of a polytope P, the *inner primal cone* of F is the cone $\mathbf{C}(F)$ generated by $\{u - v \mid u \in P, v \in F\}$, and the *outer normal cone* of F is the cone $\mathbf{C}^\diamond(F)$ generated by the outer normal vectors of the facets of P containing F. Note that these two cones are dual to one another. The *normal fan* of P is the complete polyhedral fan formed using the outer normal cones of all faces of P. See (Ziegler, 1995) for more details.

Here, we still consider a normal vector e_H to each hyperplane $H \in \mathcal{A}$ such that the base region B is contained in the positive half-space $H^+ = \{v \in V \mid \langle e_H, v\rangle \geq 0\}$. We are interested in the corresponding zonotope defined below. Details on zonotopes can be found in the book by G. M. Ziegler (Ziegler, 1995) and in the article by P. McMullen (McMullen, 1971).

Definition 4.2.19 The *zonotope* $\mathbf{Z}_{\mathcal{A}}$ of the arrangement $\mathcal{A}$ is the convex polytope

$$\mathbf{Z}_{\mathcal{A}} := \left\{ \sum_{H \in \mathcal{A}} \lambda_H e_H \;\middle|\; -1 \leq \lambda_H \leq 1 \text{ for all } H \in \mathcal{A} \right\}.$$

Example 4.2.20 The zonotope for a Coxeter arrangement is called a *permutahedron*, see (Hohlweg, 2012b). We have represented on the left of Figure 4.6 the zonotope of the arrangement of Example 4.1.1 and Figure 4.2. It has 6 vertices corresponding to the 6 regions of the arrangement, and 6 edges corresponding to the 6 rays of the arrangement.

Note that this zonotope depends upon the choices of the normal vectors e_H of the hyperplanes $H \in \mathcal{A}$, but its combinatorics does not. Namely, P. H. Edelman gives in (Edelman, 1984, Lemma 3.1) a bijection between the nonempty faces of the zonotope $\mathbf{Z}_{\mathcal{A}}$ and the the faces $\mathscr{F}_{\mathcal{A}}$ of the arrangement $\mathcal{A}$ using the τ map (given in the following lemma) which was first defined by McMullen in (McMullen, 1971, p. 92).

Lemma 4.2.21 *The map τ defined by*

$$\tau(F) = \left\{ \sum_{F \not\subseteq H} F(H) e_H + \sum_{F \subseteq H} \lambda_H e_H \;\middle|\; -1 \leq \lambda_H \leq 1 \textit{ for all } F \subseteq H \in \mathcal{A} \right\}$$

is a bijection from the faces $\mathscr{F}_{\mathcal{A}}$ to the nonempty faces of the zonotope $\mathbf{Z}_{\mathcal{A}}$. Moreover, F is the outer normal cone $\mathbf{C}^{\diamond}\big(\tau(F)\big)$ of $\tau(F)$, so that the fan of the arrangement $\mathcal{A}$ is the normal fan of $\mathbf{Z}_{\mathcal{A}}$.

We now relate the root inversion sets of § 4.2.3 to the faces of the zonotope $\mathbf{Z}_{\mathcal{A}}$.

Proposition 4.2.22 *The cone of the root inversion set $\mathbf{R}(F)$ is the inner primal cone of the face $\tau(F)$ in the zonotope $\mathbf{Z}_{\mathcal{A}}$, i.e.,*

$$\operatorname{cone}\big(\mathbf{R}(F)\big) = \mathbf{C}\big(\tau(F)\big) \quad \textit{and} \quad \mathbf{R}(F) = \mathbf{C}\big(\tau(F)\big) \cap \Phi_{\mathcal{A}}.$$

Proof. Let F be an arbitrary face in $\mathscr{F}_{\mathcal{A}}$ and let u be a point in $\mathbf{Z}_{\mathcal{A}}$. By construction we have $u = \sum_{H\in\mathcal{A}} \lambda_H e_H$ where $|\lambda_H| \leq 1$ for all $H \in \mathcal{A}$. Let v be a point in $\tau(F)$. The inner primal cone associated to F in the zonotope $\mathbf{Z}_{\mathcal{A}}$, is $\mathbf{C}\left(\tau(F)\right) = \{u - v \mid \ u \in \mathbf{Z}_{\mathcal{A}} \text{ and } v \in \tau(F)\}$.

More explicitly, if $H \in \mathcal{A}_F$ then the e_H component of $u-v$ is given by $(\lambda_H - \lambda'_H)e_H$ where $|\lambda_H| \leq 1$ and $|\lambda'_H| \leq 1$. In particular $\pm e_H \in \mathbf{C}\left(\tau(F)\right)$. If $H \notin \mathcal{A}_F$ then the component of e_H for $u-v$ is given by $(\lambda_H - \mu_H)e_H$ where $|\lambda_H| \leq 1$ and $\mu_H = \pm 1$. Recall from Lemma 4.2.21 that $\mu_H = -1$ if $F(H) = -$, etc. Suppose $\mu_H = +1$, then $-e_H \in \mathbf{C}\left(\tau(F)\right)$, but $e_H \notin \mathbf{C}\left(\tau(F)\right)$. Similarly, when $\mu_H = -1$, $e_H \in \mathbf{C}\left(\tau(F)\right)$ and $-e_H \notin \mathbf{C}\left(\tau(F)\right)$. □

Example 4.2.23 An example of the equality between the cone of the root inversion set with the inner primal cone of the face of the associated zonotope can be seen in Figure 4.6 for the type A_2 Coxeter arrangement. In Figure 4.6 we have the zonotope $\mathbf{Z}_{\mathcal{A}}$ on the left and the root inversion set for each face on the right. For a face F of $\mathcal{A}$, the cone of the root inversion set of F is the same as the inner primal cone of $\tau(F)$ in $\mathbf{Z}_{\mathcal{A}}$.

4.3 Lattice properties of the facial weak order

It was shown in (Dermenjian et al., 2018) that the facial weak order on Coxeter arrangements is a lattice. The aim of this section is to extend this result to any hyperplane arrangement with a lattice of regions.

Theorem 4.3.1 *If $\mathcal{A}$ is an arrangement where* $\mathrm{PR}(\mathcal{A}, B)$ *is a lattice, then* $\mathbf{FW}(\mathcal{A}, B)$ *is a lattice.*

In order to prove this result we use the BEZ lemma which provides a local criterion to characterize finite posets which are lattices.

Lemma 4.3.2 ((Björner et al., 1990, Lemma 2.1)) *If L is a finite, bounded poset such that the join $x \vee y$ exists whenever x and y both cover some $z \in L$, then L is a lattice.*

So the proof of Theorem 4.3.1 reduces to proving the following statement.

Theorem 4.3.3 *Let $\mathcal{A}$ be a hyperplane arrangement where* $\mathrm{PR}(\mathcal{A}, B)$ *is a lattice and let X, Y, Z be three faces of $\mathcal{A}$. If $Z \lessdot_{\mathbf{FW}} X$ and $Z \lessdot_{\mathbf{FW}} Y$, then the join $X \vee_{\mathbf{FW}} Y$ exists.*

The proof of this theorem is the aim of the next two sections. The idea of the proof is as follows. We first consider our cover relations $Z \lessdot_{\mathbf{FW}} X$ and $Z \lessdot_{\mathbf{FW}} Y$. We know from Corollary 4.2.10 that this is equivalent to $|\dim Z - \dim X| = 1$, $Z \leq_{\mathbf{FW}} X$, and either $Z \subseteq X$ or $X \subseteq Z$ and similarly for Y. By symmetry of X and Y, we thus obtain the following three cases:

(1) $X \cup Y \subseteq Z$ and $\dim X = \dim Y = \dim Z - 1$,

(2) $Z \subseteq X \cap Y$ and $\dim X = \dim Y = \dim Z + 1$, and

(3) $X \subseteq Z \subseteq Y$ and $\dim X + 1 = \dim Y - 1 = \dim Z$.

In each case we consider the subarrangement associated to the largest face contained in all three faces. For case (1) we consider the subarrangement associated to the face $X \cap Y$, $\mathcal{A}_{X \cap Y} = \{H \in \mathcal{A} \mid \ X \cap Y \subseteq H\}$, for case (2) we consider the subarrangement $\mathcal{A}_Z = \{H \in \mathcal{A} \mid \ Z \subseteq H\}$ and for case (3) we consider the subarrangement $\mathcal{A}_X = \{H \in \mathcal{A} \mid \ X \subseteq H\}$. In the next subsection we show that the join in the poset of regions of a subarrangement can be extended to a join in the poset of regions of the arrangement itself. Finally, for each case we find the join inside the appropriate subarrangement, culminating in the proof of Theorem 4.3.3.

Before we begin, we give a conjecture stating that the converse of Theorem 4.3.1 is true as well.

Conjecture 4.3.4 *For any hyperplane arrangement* $\mathcal{A}$ *and any base region* B *of* $\mathcal{A}$*, the poset of regions* $\mathrm{PR}(\mathcal{A}, B)$ *is a lattice if and only if the facial weak order* $\mathbf{FW}(\mathcal{A}, B)$ *is a lattice.*

4.3.1 Joins and subarrangements of faces

A *subarrangement* of an arrangement $\mathcal{A}$ is a subset $\mathcal{A}'$ of $\mathcal{A}$. There is a natural map $\mathscr{F}_{\mathcal{A}} \to \mathscr{F}_{\mathcal{A}'}$ that projects each face G in $\mathscr{F}_{\mathcal{A}}$ to the smallest face $G_{\mathcal{A}'}$ in $\mathscr{F}_{\mathcal{A}'}$ such that the relative interior of G is contained in the relative interior of $G_{\mathcal{A}'}$, *i.e.*, for $H \in \mathcal{A}'$ then $G_{\mathcal{A}'}(H) = G(H)$. Note that this map is surjective and preserves the facial weak order: if $F \leq_{\mathbf{FW}} G$ in $\mathcal{A}$, then $F_{\mathcal{A}'} \leq_{\mathbf{FW}} G_{\mathcal{A}'}$ in $\mathcal{A}'$.

We particularly focus on the following special subarrangements. For a face $F \in \mathscr{F}_{\mathcal{A}}$, let $\mathcal{A}_F := \{H \in \mathcal{A} \mid F \subseteq H\}$ be the subarrangement of $\mathcal{A}$ with all hyperplanes which contain F. This subarrangement $\mathcal{A}_F$ is known as the *support* of F or the *localization* of $\mathcal{A}$ to F. We denote by π_F the projection map $\mathscr{F}_{\mathcal{A}} \to \mathscr{F}_{\mathcal{A}_F}$ described above in this specific case, and we often use the shorthand G_F for $\pi_F(G) = G_{\mathcal{A}_F}$. Note that the surjection π_F restricts to a bijection between $\{G \in \mathscr{F}_{\mathcal{A}} \mid F \subseteq G\}$ and $\mathscr{F}_{\mathcal{A}_F}$.

Example 4.3.5 Figure 4.7 gives an example of these maps for the subarrangement $\mathcal{A}_{F_1}$ of the type A_2 arrangement discussed in Example 4.1.1. Since H_2 is the only hyperplane containing F_1, our subarrangement contains one hyperplane $\mathcal{A}_{F_1} = \{H_2\}$. Then $\pi_{F_1} : \mathscr{F}_{\mathcal{A}} \to \mathscr{F}_{\mathcal{A}_{F_1}}$ is the map with the following equalities (some of which are shown in the figure, but not all).

$$\pi_{F_1}(R_2) = \pi_{F_1}(F_2) = \pi_{F_1}(R_3) = \pi_{F_1}(F_3) = \pi_{F_1}(R_4)$$
$$\pi_{F_1}(F_1) = \pi_{F_1}(0) = \pi_{F_1}(F_4)$$
$$\pi_{F_1}(R_1) = \pi_{F_1}(F_0) = \pi_{F_1}(B) = \pi_{F_1}(F_5) = \pi_{F_1}(R_5)$$

It can be seen in the figure that π_{F_1} is a bijection from $\{R_1, R_2, F_1\}$ to $\mathcal{A}_{F_1}$.

Given an arrangement $\mathcal{A}$ whose poset of regions $\mathrm{PR}(\mathcal{A}, B)$ is a lattice, it is not necessary that any arbitrary subarrangement will also have a lattice of regions. However, when the subarrangement is associated to a face, then the lattice property of the poset of regions is preserved through facial intervals. This follows from the well-known fact that an interval of a lattice is a lattice. This lattice property, combined with the fact that the base region of a lattice of regions is always simplicial (see (Edelman, 1984, Theorem 3.1 and 3.4)) gives the following proposition.

Proposition 4.3.6 *Let $\mathcal{A}$ be an arrangement whose poset of regions $\mathrm{PR}(\mathcal{A}, B)$ is a lattice. For a face $F \in \mathscr{F}_{\mathcal{A}}$ the subarrangement $\mathcal{A}_F$ is a central subarrangement and $\mathrm{PR}(\mathcal{A}_F, B_F)$ is a lattice of regions with simplicial base region B_F.*

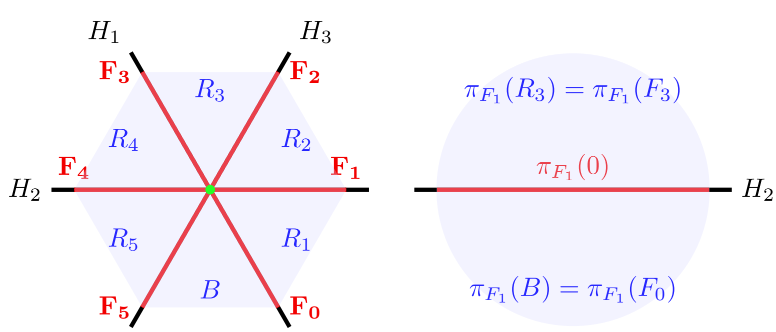


Figure 4.7: The map π_{F_1} from an arrangement $\mathcal{A}$ to a subarrangement $\mathcal{A}_{F_1}$. See Example 4.3.5.

Lemma 4.3.7 *For any three faces $X, Y, Z \in \mathscr{F}_{\mathcal{A}}$ such that $[X, Y]$ is an interval in $\mathbf{FW}(\mathcal{A}, B)$ and $Z \subseteq X \cap Y$, then the interval $[X, Y]$ in $\mathbf{FW}(\mathcal{A}, B)$ is isomorphic to $[X_Z, Y_Z]$ in $\mathbf{FW}(\mathcal{A}_Z, B_Z)$.*

Proof. We first prove that the map $W \mapsto W_Z$ defines an injective order preserving map from $[X, Y]$ to $[X_Z, Y_Z]$. Let $W \in [X, Y]$. We aim to show Z is a face of W. By Proposition 4.2.4 it suffices to show $Z(H) = W(H)$ when $Z(H) \neq 0$. Suppose that there is $H \in \mathcal{A}$ such that $Z(H) \neq W(H)$ and $Z(H) \neq 0$. This implies $Z(H) = X(H) = Y(H)$. As $W \in [X, Y]$, then $Y(H) \leq W(H) \leq X(H)$. Hence $W(H) = X(H) = Z(H)$, a contradiction. Therefore, Z is a face of W. Thus, the localization map $[X, Y] \to [X_Z, Y_Z]$ is injective.

The inverse map $[X_Z, Y_Z] \to [X, Y]$ is defined by extending W_Z to a covector W with $W(H) = Z(H)$ for $H \in \mathcal{A} \setminus \mathcal{A}_Z$. As this map is also order preserving, the proof is complete. □

This lemma gives us another way to view facial intervals as the faces of a subarrangement. With the above lemma, given a facial interval $[m_F, M_F]$ for a face F, then m_F (resp. M_F) is the region in $\mathcal{A}$ associated to the base region B_F (resp. to its opposite region $-B_F$) in $\mathscr{F}_{\mathcal{A}_F}$. We now show that a join in the poset of regions of a subarrangement extends to a join in the poset of regions of the arrangement itself. The following is possible by Proposition 4.3.6.

Proposition 4.3.8 *For any three faces $X, Y, Z \in \mathscr{F}_{\mathcal{A}}$ such that $Z \subseteq X \cap Y$, if there exists a face W containing Z such that $W_Z = X_Z \vee_{\mathbf{FW}} Y_Z$ in $\mathbf{FW}(\mathcal{A}_Z, B_Z)$ then $W = X \vee_{\mathbf{FW}} Y$ in $\mathbf{FW}(\mathcal{A}, B)$.*

Proof. Suppose $U \in \mathscr{F}_{\mathcal{A}}$ is a face such that $X \leq_{\mathbf{FW}} U$ and $Y \leq_{\mathbf{FW}} U$. Since the projection map $\pi_F : \mathscr{F}_{\mathcal{A}} \to \mathscr{F}_{\mathcal{A}_F}$ preserves the facial weak order, we have that $W_Z = X_Z \vee_{\mathbf{FW}} Y_Z \leq_{\mathbf{FW}} U_Z$ in the facial weak order of the subarrangement.

In other words, for all $H \in \mathcal{A}_Z$, we have $U_Z(H) \leq W_Z(H)$, and thus $U(H) \leq W(H)$.

Next let H' be a hyperplane in $\mathcal{A} \smallsetminus \mathcal{A}_Z$. Since $H' \notin \mathcal{A}_Z$, we have $Z(H') \neq 0$. Furthermore, by Proposition 4.2.4, $0 \neq Z(H') = X(H') = Y(H') = W(H')$ since $Z \subseteq X \cap Y$ and $Z \subseteq W$. Then, since $X \leq_{\mathbf{FW}} U$, we have $U(H') \leq X(H') = W(H')$.

In other words, $U(H) \leq W(H)$ for all $H \in \mathcal{A}$. Therefore $W \leq_{\mathbf{FW}} U$ implying $W = X \vee_{\mathbf{FW}} Y$. □

4.3.2 Joins in subarrangements

As discussed, we now describe the three distinct cases that arise using the cover relations of the facial weak order. Then, for each case, we restrict ourselves to the subarrangement associated to the largest face contained in all three faces and find the join in the subarrangement. Combining these results with Proposition 4.3.8 proves Theorem 4.3.3.

Consider three faces $X, Y, Z \in \mathscr{F}_{\mathcal{A}}$ such that $Z \lessdot_{\mathbf{FW}} X$ and $Z \lessdot_{\mathbf{FW}} Y$. Recall that by Corollary 4.2.10, we have $Z \lessdot_{\mathbf{FW}} X$ if and only if $|\dim Z - \dim X| = 1$, $Z \leq_{\mathbf{FW}} X$, and either $Z \subseteq X$ or $X \subseteq Z$, and similarly for Y. By symmetry on X and Y, this gives us three different cases:

(1) $X \cup Y \subseteq Z$ and $\dim X = \dim Y = \dim Z - 1$,

(2) $Z \subseteq X \cap Y$ and $\dim X = \dim Y = \dim Z + 1$, and

(3) $X \subseteq Z \subseteq Y$ and $\dim X + 1 = \dim Y - 1 = \dim Z$.

We now look at each case individually. We have broken down their proofs into three subsections to better facilitate their reading. We let $\mathscr{B}(R)$ denote the set of boundary hyperplanes of a region R.

First case: $X \cup Y \subseteq Z$ and $\dim X = \dim Y = \dim Z - 1$

Since $X \cap Y$ is the largest face contained in X, Y and Z, we restrict to the subarrangement $\mathcal{A}_{X\cap Y}$ and find the join there. An example (in rank 2) is given in Figure 4.8. By Proposition 4.3.6, the poset of regions $\mathrm{PR}(\mathcal{A}_{X\cap Y}, B_{X\cap Y})$ is a lattice. Thus, without loss of generality, it suffices to prove the following proposition.

Proposition 4.3.9 *Consider an arrangement $\mathcal{A}$ whose poset of regions is a lattice with three faces X, Y and Z such that $Z \lessdot_{\mathbf{FW}} X$, $Z \lessdot_{\mathbf{FW}} Y$, $\{0\} = X \cap Y$ and $X \cup Y \subseteq Z$. Then $\{0\} = X \cap Y = X \vee_{\mathbf{FW}} Y$.*

Proof. We first prove that $X \leq_{\mathbf{FW}} X \cap Y = \{0\}$. Assume by contradiction that there is $H \in \mathcal{A}$ such that $X(H) = -$. Since $X \subseteq Z$, we obtain that $Z(H) = -$ by Proposition 4.2.4. Moreover, since $Z \lessdot_{\mathbf{FW}} Y$, we have $Y(H) \leq Z(H) = -$.

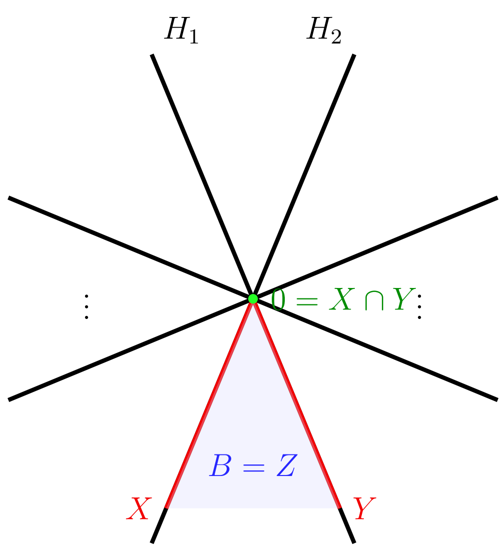


Figure 4.8: The construction of the join for the first case when $X \cup Y \subseteq Z$.

Let $[m_Z, M_Z]$ be the facial interval in the poset of regions associated to Z. As $\dim X = \dim Y = \dim Z - 1$, there exist boundary hyperplanes H_1 and H_2 of m_Z such that $X = Z \cap H_1$ and $Y = Z \cap H_2$. Since $X \neq Z \neq Y$, we obtain by Lemma 4.2.7 that $Z(H_1) \neq 0 \neq Z(H_2)$. Since $0 = X(H_1) \leq Z(H_1)$ and since $0 = Y(H_2) \leq Z(H_2)$, then we conclude that $Z(H_1) = Y(H_1) = +$ and $Z(H_2) = X(H_2) = +$.

Let $\mathcal{A}' := \{H_1, H_2, H\}$ be the subarrangement of $\mathcal{A}$ with these three hyperplanes. Since $Z(H_1) = Z(H_2) = +$ and $Z(H) = -$, the face $Z_{\mathcal{A}'}$ is a region in $\mathcal{A}'$. Moreover, we have $(X \cap Y)_{\mathcal{A}'}(H) = 0$ because $(X \cap Y)(H) = 0$. We thus obtain that H_1 and H_2 are the only boundary hyperplanes of $Z_{\mathcal{A}'}$ in $\mathcal{A}'$. Therefore, for any region $R \in \mathscr{R}_{\mathcal{A}'} \setminus \{Z_{\mathcal{A}'}\}$, then either H_1 or H_2 is in the separation set $S(R, Z_{\mathcal{A}'})$. Since $Z_{\mathcal{A}'}(H_1) = Z_{\mathcal{A}'}(H_2) = +$, then either $R(H_1) = -$ or $R(H_2) = -$. It follows that no region of $\mathcal{A}'$ is all positive, a contradiction since the base region $B_{\mathcal{A}'}$ is all positive.

We conclude that $X(H) \geq 0$ for all $H \in \mathcal{A}$, so that $X \leq_{\mathbf{FW}} X \cap Y = \{0\}$. By symmetry, we also obtain that $Y \leq_{\mathbf{FW}} X \cap Y = \{0\}$.

Finally, to prove that $\{0\} = X \cap Y = X \vee_{\mathbf{FW}} Y$, we consider an arbitrary face U in $\mathcal{A}$ such that $X \leq_{\mathbf{FW}} U$ and $Y \leq_{\mathbf{FW}} U$. Then, $U(H) \leq \min\big(X(H), Y(H)\big)$ for all $H \in \mathcal{A}$. Since $(X \cap Y)(H) = 0 \leq \min\big(X(H), Y(H)\big)$ for all $H \in \mathcal{A}$, the faces X and Y of $\mathscr{F}_{\mathcal{A}}$ are contained in B by Corollary 4.2.6. By Proposition 4.3.6, B is simplicial and therefore, there exists H_3, H_4 in $\mathscr{B}(B)$ such that $X \cap Y = X \cap H_3 = Y \cap H_4$ with $X(H) = Y(H) = 0$ for all $H \in \mathscr{B}(B) \setminus \{H_3, H_4\}$. Note that H_3 and H_4 could be the same H_1 and H_2 as before. Therefore, $0 = \min\big(X(H), Y(H)\big)$ for all $H \in \mathscr{B}(B)$. We conclude that $U(H) \leq 0$ for all $H \in \mathscr{B}(B)$. By Corollary 4.2.6, $U(H) \leq 0$ for all $H \in \mathcal{A}$. Therefore, $X \cap Y \leq_{\mathbf{FW}} U$. □

Second case: $Z \subseteq X \cap Y$ and $\dim X = \dim Y = \dim Z + 1$

Since Z is the largest face contained in X, Y and Z, we restrict to the subarrangement $\mathcal{A}_Z$ and find the join there. An example (in rank 2) is given in Figure 4.9. By Proposition 4.3.6, the poset of regions $\mathrm{PR}(\mathcal{A}_Z, B_Z)$ is a lattice. Therefore, without loss of generality, we consider an arrangement $\mathcal{A}$ whose poset of regions is a lattice with distinct faces X, Y, and $Z = \{0\}$ such that $\{0\} = Z \lessdot_{\mathbf{FW}} X$ and $\{0\} = Z \lessdot_{\mathbf{FW}} Y$. Observe that this implies by Corollary 4.2.6 that X and Y are rays of the region $-B$ opposite to the base region B. Since $\mathrm{PR}(\mathcal{A}, B)$ is a lattice, $-B$ is simplicial, and therefore there is a 2-dimensional face W of $-B$ containing both X and Y. This gives us the join of X and Y.

Proposition 4.3.10 *Consider an arrangement $\mathcal{A}$ whose poset of regions is a lattice with distinct faces X, Y, and $Z = \{0\}$ such that $\{0\} = Z \lessdot_{\mathbf{FW}} X$ and $\{0\} = Z \lessdot_{\mathbf{FW}} Y$. Then $X \vee_{\mathbf{FW}} Y = W$ where W is the 2-dimensional face of $-B$ containing both X and Y.*

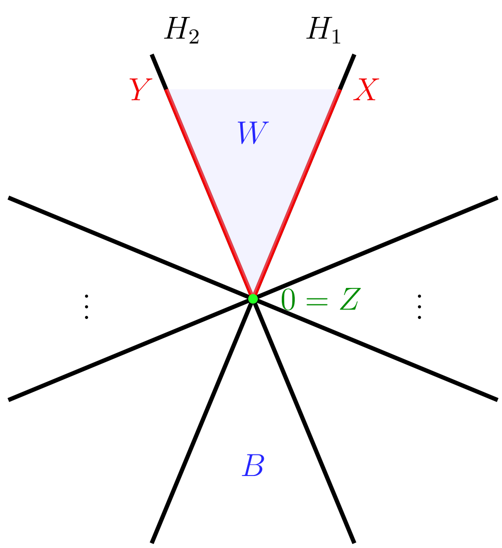


Figure 4.9: The construction of the join for the second case when $Z \subseteq X \cap Y$.

Proof. Since $X \subseteq W$ are all faces of $-B$, we have $X \leq_{\mathbf{FW}} W$ by Corollary 4.2.14. Similarly, we have $Y \leq_{\mathbf{FW}} W$.

Conversely, consider a face $U \in \mathscr{F}_{\mathcal{A}}$ such that $X \leq_{\mathbf{FW}} U$ and $Y \leq_{\mathbf{FW}} U$. For any $H \in \mathcal{A}$, we have $X(H) \leq 0$ by Corollary 4.2.6 since X is a face of $-B$. Since $X \leq_{\mathbf{FW}} U$, we have $U(H) \leq X(H) \leq 0$ for all $H \in \mathcal{A}$, which implies that U is a face of $-B$ by Corollary 4.2.6. Since $X \leq_{\mathbf{FW}} U$, Corollary 4.2.14 implies that $X \subseteq U$. Similarly, $Y \subseteq U$ and thus $W \subseteq U$. We conclude that $W \leq_{\mathbf{FW}} U$ by Corollary 4.2.14. □

Third case: $X \subseteq Z \subseteq Y$ and $\dim X + 1 = \dim Y - 1 = \dim Z$

Since X is the largest face contained in X, Y and Z, we restrict to the subarrangement $\mathcal{A}_X$ and find the join there. An example (in rank 2) is given in Figure 4.10. By Proposition 4.3.6, the poset of regions $\mathrm{PR}(\mathcal{A}_X, B_X)$ is a lattice. Therefore, without loss of generality, we consider an arrangement $\mathcal{A}$ whose poset of regions

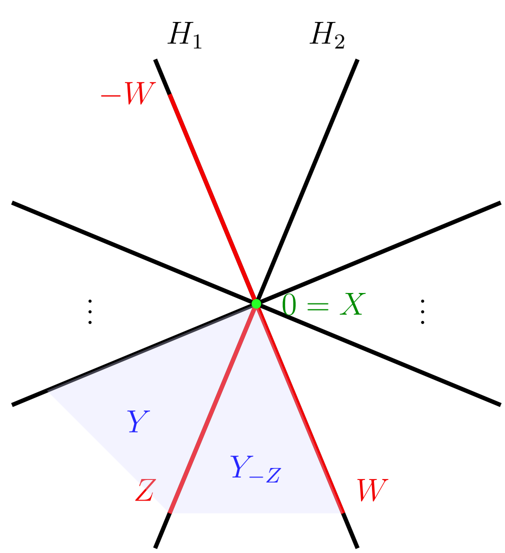


Figure 4.10: The construction of the join in the third case when $X \subseteq Z \subseteq Y$.

is a lattice with three faces $X = \{0\}$, Y and Z such that $Z \lessdot_{\mathbf{FW}} X = \{0\}$, $Z \lessdot_{\mathbf{FW}} Y$ and $\{0\} = X \subseteq Z \subseteq Y$. Observe that this implies that Z is a ray of the base region B by Corollary 4.2.6. Remember that for two faces F and G, we denote by F_{-G} the reorientation of F by G (see § 4.2.1). Observe that Z is a ray of the 2-dimensional cone Y_{-Z}, and let W denote its other ray. We aim to prove the following proposition.

Proposition 4.3.11 *Let $\mathcal{A}$ be an arrangement with a lattice of regions and three faces $X = \{0\}$, Y and Z such that $Z \lessdot_{\mathbf{FW}} X = \{0\}$, $Z \lessdot_{\mathbf{FW}} Y$ and $\{0\} = X \subseteq Z \subseteq Y$. Then $X \vee_{\mathbf{FW}} Y = -W$ where W is the ray of Y_{-Z} distinct from Z.*

We will prove that $X \vee_{\mathbf{FW}} Y = -W$ in Lemma 4.3.14 and Lemma 4.3.15. We first identify two crucial boundary hyperplanes of the base region B.

Lemma 4.3.12 *There exists two unique boundary hyperplanes H_1 and H_2 of the base region B such that $\{0\} = X = Z \cap H_1$ and $Z = Y \cap H_2$.*

Proof. As Z is a ray of the (simplicial) base region B, there is a unique $H_1 \in \mathscr{B}(B)$ such that $\{0\} = X = Z \cap H_1$. For the second hyperplane, we first claim that there is a unique boundary hyperplane H_2 of the base region such that $Y(H_2) = -$ while $Z(H_2) = 0$. Indeed, if there were two such hyperplanes H_2 and H_2', we would have $Z \subsetneq Y \cap H_2 \subsetneq Y$ contradicting that $\dim Y = \dim Z + 1$. Moreover, since $Z \leq_{\mathbf{FW}} Y$, there is no hyperplane H such that $Y(H) = +$ and $Z(H) = 0$. We conclude that H_2 is the unique hyperplane of $\mathscr{B}(B)$ such that $Z = Y \cap H_2$. $\square$

Lemma 4.3.13 *Consider the two boundary hyperplanes H_1 and H_2 of the base region given in Lemma 4.3.12. Then*

$$0 = W(H_1) = X(H_1) < Y(H_1) = Z(H_1) = +,$$

$$- = Y(H_2) < X(H_2) = 0 = Z(H_2) < W(H_2) = +, \text{ and}$$

$$0 = W(H) = X(H) = Y(H) = Z(H) \text{ for all } H \in \mathscr{B}(B) \smallsetminus \{H_1, H_2\}.$$

Proof. Since $X = \{0\}$, we have $X(H) = 0$ for all $H \in \mathscr{B}(B)$. By definition of H_1 and H_2 and Lemma 4.2.7, we have $X(H_1) = 0 \neq Z(H_1)$ and $Z(H_2) = 0 \neq Y(H_2)$. Since $Z \leq_{\mathbf{FW}} X$ and $Z \leq_{\mathbf{FW}} Y$, this implies that $Z(H_1) = +$ and $Y(H_2) = -$. Moreover, as Z is a face of Y, we obtain that $Y(H_1) = +$. Finally, for any hyperplane $H \in \mathscr{B}(B) \smallsetminus \{H_1, H_2\}$, we have $Z(H) = 0$ by uniqueness of H_2, and therefore $Y(H) = 0$ since $\dim Y = \dim Z + 1$ and $Z = Y \cap H_2$.

By definition of the reorientation operation, we thus obtain that $Y_{-Z}(H_1) = +$ and $Y_{-Z}(H_2) = +$, while $Y_{-Z}(H) = 0$ for all $H \in \mathscr{B}(B) \smallsetminus \{H_1, H_2\}$. In other words, Y_{-Z} is the 2-dimensional face of the base region B given by its intersection with all hyperplanes of $\mathscr{B}(B) \smallsetminus \{H_1, H_2\}$. Finally, since W is the ray of Y_{-Z} distinct from Z, we obtain that $W(H_1) = 0$, that $W(H_2) = +$ and that $W(H) = 0$ for all $H \in \mathscr{B}(B) \smallsetminus \{H_1, H_2\}$. □

Lemma 4.3.14 *We have $X \lessdot_{\mathbf{FW}} -W$ and $Y \leq_{\mathbf{FW}} -W$ in the facial weak order.*

Proof. By Lemma 4.3.13, $W(H) \geq 0$ for all $H \in \mathscr{B}(B)$, therefore $W(H) \geq 0$ for all $H \in \mathcal{A}$ by Corollary 4.2.6. We therefore obtain that both W and Z are rays of the base region B, and thus Y_{-Z} is a 2-dimensional face of B as well.

Since $X = \{0\}$ and W is a ray of the base region B, we have that $-W(H) \leq X(H)$ for any $H \in \mathcal{A}$ so that $X \leq_{\mathbf{FW}} -W$. Since $X \subseteq -W$ and $\dim(-W) - \dim(X) = 1$, we obtain that $X \lessdot_{\mathbf{FW}} -W$ by Proposition 4.1.13.

Assume now by contradiction that $Y \not\leq_{\mathbf{FW}} -W$. Then there exists $H \in \mathcal{A}$ such that $Y(H) < -W(H)$. Since $W(H) \geq 0$, it implies that $Y(H) = -$ and $W(H) = 0$. But since $Y_{-Z}(H) \geq 0$, we obtain by definition of reorientation that $Z(H) = 0$ and $Y_{-Z}(H) = +$. We conclude that $W(H) = Z(H) = 0$ while $Y_{-Z} = +$, contradicting the fact that Y_{-Z} is the 2-dimensional face with rays W and Z. □

Lemma 4.3.15 *We have $X \vee_{\mathbf{FW}} Y = -W$.*

Proof. Consider a face U of $\mathscr{F}_{\mathcal{A}}$ such that $X \leq_{\mathbf{FW}} U$ and $Y \leq_{\mathbf{FW}} U$. We have that $U(H) \leq X(H) = 0$ for all $H \in \mathcal{A}$ and, moreover, $U(H_2) \leq Y(H_2) = -$. Therefore, we obtain that U is a face of $-B$ and $-W \subseteq U$. We conclude that $-W \leq_{\mathbf{FW}} U$ by Corollary 4.2.14. □

4.3.3 Further lattice properties of the facial weak order

We end this section by describing some lattice properties of the facial weak order. In particular we show that the lattice is self-dual, show the poset of regions is a sublattice, describe all the join-irreducible elements and show semidistributivity.

Duality

Recall that the *dual* of a lattice $(L, \leq)$ is the order $(L, \leq^{\text{op}})$ where for $u, v \in L$, we have $u \leq v$ if and only if $v \leq^{\text{op}} u$. A lattice is *self-dual* if it is isomorphic to its dual. As with the poset of regions, the facial weak order is self-dual. This follows from the fact that the poset of regions is itself self-dual and from the fact that the negative of every covector must also be in the set of covectors by the definition of oriented matroid.

Proposition 4.3.16 *The map $F \mapsto -F := \{-v \mid v \in F\}$ is a self-duality of the facial weak order $\mathbf{FW}(\mathcal{A}, B)$.*

Sublattice

In this subsection we show that if $\mathcal{A}$ is simplicial, not only is $\mathrm{PR}(\mathcal{A}, B)$ an induced subposet of $\mathbf{FW}(\mathcal{A}, B)$ by Remark 4.1.12 and a lattice by Theorem 4.1.3, but it is in fact a sublattice of $\mathbf{FW}(\mathcal{A}, B)$. Recall that a *sublattice* L' of a lattice L is an induced subposet such that $u \vee v \in L'$ and $u \wedge v \in L'$ for any $u, v \in L'$. The proof requires the following lemma which, just like the BEZ lemma, gives us a local way to verify if a subposet is a sublattice of a lattice, see (Reading, 2016, Lemma 9-2.11). Recall that a poset is connected when the transitive closure of its comparability relation forms a single equivalence class.

Lemma 4.3.17 *If P is a connected finite induced subposet of a lattice L such that $x \vee y \in P$ for all $x, y, z \in P$ with $z \lessdot x$ and $z \lessdot y$, and $x \wedge y \in P$ for all $x, y, z \in P$ with $x \lessdot z$ and $y \lessdot z$, then P is a sublattice of L.*

With this tool, we can prove the following statement.

Proposition 4.3.18 *For a simplicial arrangement $\mathcal{A}$, the lattice of regions is a sublattice of the facial weak order $\mathbf{FW}(\mathcal{A}, B)$.*

Proof. By Remark 4.1.12, $\mathrm{PR}(\mathcal{A}, B)$ is an induced subposet of $\mathbf{FW}(\mathcal{A}, B)$. It is clearly connected as it contains the minimal and maximal elements of $\mathbf{FW}(\mathcal{A}, B)$. Finally, by Proposition 4.3.16, we just need to prove one of the two criteria of

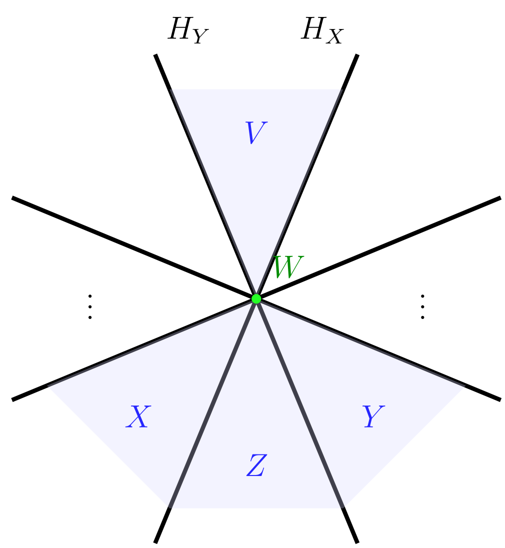


Figure 4.11: The construction of the join when X and Y are regions.

Lemma 4.3.17. Consider thus three distinct regions $X, Y, Z \in \mathscr{R}_{\mathcal{A}}$ such that $Z \lessdot_{\mathrm{PR}} X$ and $Z \lessdot_{\mathrm{PR}} Y$. See Figure 4.11 for a (rank 2) example. Since $Z \lessdot_{\mathrm{PR}} X$, there is a hyperplane H_X separating X and Z such that $S(X) = S(Z) \cup \{H_X\}$. Similarly, there is a hyperplane H_Y separating Y and Z such that $S(Y) = S(Z) \cup \{H_Y\}$. Since Z is simplicial, the face $W := Z \cap H_X \cap H_Y$ has codimension 2. We thus consider the rank 2 subarrangement $\mathcal{A}_W$. Since $Z(H_X) = Z(H_Y) = +$, the face Z_W is the base region of $\mathcal{A}_W$. Moreover, since we have $X_W(H_X) = X(H_X) = -$ and we have $Y_W(H_Y) = Y(H_Y) = -$, the join V of X_W and Y_W in $\mathcal{A}_W$ satisfies $V(H_X) = V(H_Y) = -$ and is thus the opposite of the base region in $\mathcal{A}_W$. By Proposition 4.3.8, the join of X and Y is the face U of $\mathcal{A}$ containing W and such that $U_W = V$. We conclude that $X \vee Y$ is full dimensional, thus is a region. This concludes the proof by Lemma 4.3.17 and Proposition 4.3.16. □

Join-irreducible elements

We next aim to find all the join-irreducible elements of the facial weak order. An element x of a finite lattice L is *join-irreducible* if $x \neq \bigvee L'$ for all $L' \subseteq L \smallsetminus \{x\}$. Equivalently, x is join-irreducible if and only if it covers exactly one element $x_\star$ of L. A *meet-irreducible* element y is defined in a similar manner where $y^\star$ is the unique element covering y.

For ease of notation, we denote by JIrr(**FW**) and JIrr(PR) (resp. MIrr(**FW**) and MIrr(PR)) the sets of join-irreducible (resp. meet-irreducible) elements in the facial weak order and in the poset of regions.

It turns out that the join-irreducible elements of the facial weak order are characterized by the join-irreducible elements of the poset of regions. Each region $R \in$ JIrr(PR) gives a join-irreducible face R in the facial weak order. Additionally, the facet between R and the unique region $R_\star$ it covers in the poset of regions is also a join-irreducible element in the facial weak order. We give a small lemma before characterizing the join-irreducible elements in the facial weak order of a simplicial arrangement.

Lemma 4.3.19 *Suppose $\mathcal{A}$ is a simplicial hyperplane arrangement and F a face of the arrangement. There exists exactly* $\operatorname{codim}(F)$ *facets of F weakly below F in the facial weak order.*

Proof. If F is a codimension $\operatorname{codim}(F)$ face then its span is the intersection of at least $\operatorname{codim}(F)$ hyperplanes. Since $\mathcal{A}$ is simplicial, exactly $\operatorname{codim}(F)$ of these hyperplanes bound the base region of $\mathcal{A}_F$. Let $\mathcal{H}$ be this set of bounding hyperplanes. For each $H \in \mathcal{H}$ there exists a unique face G such that $G(H) = +$ and $G(H') = 0$ for all $H' \in \mathcal{H} \smallsetminus \{H\}$ since the base region must be simplicial by Proposition 4.3.6. In other words, there exists exactly $\operatorname{codim}(F)$ many codimension $\operatorname{codim}(F) - 1$ faces covered by F in the facial weak order. □

Proposition 4.3.20 *Suppose $\mathcal{A}$ is a simplicial hyperplane arrangement and let F be a face with associated facial interval $[m_F, M_F]$. Then $F \in \mathrm{JIrr}(\mathbf{FW})$ if and only if $M_F \in \mathrm{JIrr}(\mathrm{PR})$ and $\operatorname{codim}(F) \in \{0, 1\}$.*

Proof. We first suppose that F is join-irreducible in $\mathbf{FW}(\mathcal{A}, B)$. Since a join-irreducible element can cover at most one element, Lemma 4.3.19 implies $\operatorname{codim}(F) \leq 1$.

Suppose first that $\operatorname{codim}(F) = 0$. Then F is a region and $m_F = M_F = F$. Let $F_\star$ be the unique face covered by F. By Corollary 4.2.10, $|\dim(F) - \dim(F_\star)| = 1$ and therefore $\operatorname{codim}(F_\star) = 1$. Therefore, there exists a unique hyperplane H bounding M_F such that $H \in S(M_F)$ and $H \cap M_F = F_\star$. Thus, there is a unique region R such that $S(R) = S(M_F) \smallsetminus \{H\}$. In other words, $M_F \in \mathrm{JIrr}(\mathrm{PR})$.

Suppose next that $\operatorname{codim}(F) = 1$. Again by Corollary 4.2.10 only codimension 0 and codimension 2 faces can be covered by F in the facial weak order. By

Lemma 4.3.19 there exists at least one codimension 0 face covered by F. Therefore $F_\star$ is a region and, as F is join-irreducible, F does not cover any codimension 2 face. If contrarily $M_F \notin \mathrm{JIrr}(\mathrm{PR})$ then there exists a boundary hyperplane H of M_F such that $H \cap M_F \neq F$. Let $G = H \cap M_F$. Then $G \cap F$ is a face with codimension 2 such that $G \cap F \subseteq F$ and $M_F = M_G$. Thus $G \cap F$ is a codimension 2 face covered by F, a contradiction.

To show the other direction, we conversely suppose that $M_F \in \mathrm{JIrr}(\mathrm{PR})$ and $\mathrm{codim}(F) \in \{0, 1\}$. Since $M_F \in \mathrm{JIrr}(\mathrm{PR})$ it covers the unique region $M_{F_\star}$ and there is a unique face G between the two regions with facial interval $[M_{F_\star}, M_F]$. If $\mathrm{codim}(F) = 0$ then $F = M_F$ and, since only codimension 1 faces can be covered by F, then G is the unique facet of F which is covered by F, *i.e.*, $F \in \mathrm{JIrr}(\mathbf{FW})$. If $\mathrm{codim}(F) = 1$ then $F = G$ and $M_{F_\star} \lessdot_{\mathbf{FW}} F$ by construction. To prove F doesn't cover another face it suffices to observe that if there was another face G' covered by F it must be of codimension 2 by Lemma 4.3.19. But then, $M_F \cap G' = G'$ since $G' \subseteq F \subseteq M_F$. In other words, there exists a second facet to M_F weakly below M_F by simpliciality, a contradiction. □

As we saw previously, these join-irreducibles come in pairs. This comes from introducing the edges of the poset of regions as vertices in the facial weak order.

Corollary 4.3.21 *Let F and F' be faces of codimension 1 and 0 respectively such that $F \lessdot_{\mathbf{FW}} F'$. Then F is join-irreducible in the facial weak order if and only if F' is join-irreducible in the facial weak order.*

Proof. If F is a codimension 1 face then there exists a unique codimension 0 face covering it. Conversely, for every codimension 0 face (excluding the base region) there is at least one codimension 1 face covered by it. In other words, F exists if and only if F' exists (where F' is not the base region). Furthermore, $F \lessdot_{\mathbf{FW}} F'$ implies F is the face strictly below the region F'. In other words,

$M_F = M_{F'}$. But this implies $M_F \in \mathrm{JIrr}(\mathrm{PR})$ if and only if $M_{F'} \in \mathrm{JIrr}(\mathrm{PR})$. Since $\mathrm{codim}(F) = 1$ and $\mathrm{codim}(F') = 0$ this implies F is join-irreducible if and only if F' is join-irreducible. $\square$

Recalling that our lattice is self-dual by Proposition 4.3.16 we have the following two corollaries.

Corollary 4.3.22 *Suppose $\mathcal{A}$ is a simplicial arrangement and let F be a face with associated facial interval $[m_F, M_F]$. Then $F \in \mathrm{MIrr}(\mathbf{FW})$ if and only if $m_F \in \mathrm{MIrr}(\mathrm{PR})$ and $\mathrm{codim}(F) \in \{0, 1\}$.*

Corollary 4.3.23 *Let F and F' be faces of codimension 0 and 1 respectively such that $F \lessdot_{\mathbf{FW}} F'$. Then F is meet-irreducible in the facial weak order if and only if F' is meet-irreducible in the facial weak order.*

Semidistributivity

In this subsection, we show that our lattice is semidistributive. A lattice is *join-semidistributive* if $x \vee y = x \vee z$ implies $x \vee y = x \vee (y \wedge z)$. Similarly, a lattice is *meet-semidistributive* if the dual condition holds. A lattice is *semidistributive* if it is both meet-semidistributive and join-semidistributive.

Recall that for a join-irreducible element x, the unique element it covers is denoted by $x_\star$, *i.e.*, $x_\star \lessdot x$. Likewise, for a meet-irreducible element y, the unique element covered by it is denoted $y^\star$, *i.e.*, $y \lessdot y^\star$. Given a join-irreducible element x and a meet-irreducible element y for a finite lattice L, we say that $(x_\star, x)$ and $(y, y^\star)$ are *perspective* if $x \wedge y = x_\star$ and $x \vee y = y^\star$. We have the following lemma, see (Freese et al., 1995, Theorem 2.56).

Lemma 4.3.24 *A finite lattice is meet-semidistributive if and only if for every join-irreducible element x there exists a unique meet-irreducible element y such that $(x_\star, x)$ and $(y, y^\star)$ are perspective.*

We will use the following theorem, see (Reading, 2003, Theorem 3).

Theorem 4.3.25 *For a simplicial arrangement $\mathcal{A}$, its poset of regions is a semidistributive lattice.*

It turns out that due to the self-duality of the facial weak order and its intimate connection with the poset of regions that perspective pairs do exist. This will give us that the facial weak order is semidistributive. Recall that JIrr(**FW**) (JIrr(PR)) and MIrr(**FW**) (MIrr(PR)) are the sets of join-irreducible and meet-irreducible elements in the facial weak order (poset of regions) respectively.

Proposition 4.3.26 *The facial weak order is meet-semi-distributive.*

Proof. By Lemma 4.3.24 it suffices to show for every face $F \in$ JIrr(**FW**) there exists a unique face $G \in$ MIrr(**FW**) such that $(F_\star, F)$ and $(G, G^\star)$ are perspective, *i.e.*, $F \wedge_{\mathbf{FW}} G = F_\star$ and $F \vee_{\mathbf{FW}} G = G^\star$. An example can be seen in Figure 4.12. Let $[m_F, M_F]$ denote the facial interval of F. Since $F \in$ JIrr(**FW**), by Proposition 4.3.20, $M_F \in$ JIrr(PR) and $\operatorname{codim} F \in \{0, 1\}$.

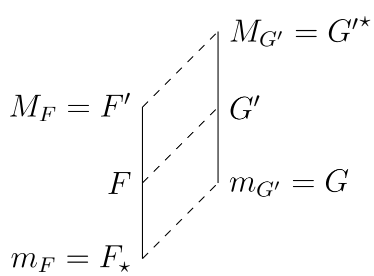


Figure 4.12: Meet-semidistributivity in the facial weak order.

Suppose $\operatorname{codim}(F) = 1$. Since $F \in \mathrm{JIrr}(\mathbf{FW})$ by Lemma 4.3.19 there exists a unique face F' of codimension 0 such that $F \lessdot_{\mathbf{FW}} F'$ and $M_F = M_{F'}$. By Corollary 4.3.21 $F' \in \mathrm{JIrr}(\mathbf{FW})$ with $F = F'_\star$. We then have the following chain of covers $F_\star \lessdot_{\mathbf{FW}} F = F'_\star \lessdot_{\mathbf{FW}} F'$.

Since $M_{F'} = M_F \in \mathrm{JIrr}(\mathrm{PR})$, by Theorem 4.3.25 and Lemma 4.3.24, there exists a unique meet-irreducible region M_G such that $((M_{F'})_\star, M_{F'})$ and $(M_G, (M_G)^\star)$ are perspective in the poset of regions. Let $G = M_G$ be the codimension 0 face associated to the region M_G. Since $M_G = m_G$ is meet-irreducible in the poset of regions, then G is meet-irreducible in the facial weak order by Corollary 4.3.22 since it is of codimension 0. Then, by definition of meet-irreducible, there exists a unique face G' of codimension 1 such that $G \lessdot_{\mathbf{FW}} G'$. Furthermore, by Corollary 4.3.23, G' is meet-irreducible in the facial weak order with $m_G = m_{G'}$. We then have the following chain of covers $G \lessdot_{\mathbf{FW}} G^\star = G' \lessdot_{\mathbf{FW}} G'^\star$.

Recalling that $((M_{F'})_\star, M_{F'})$ and $(M_G, (M_G)^\star)$ are perspective in the poset of regions, and furthermore, since $M_{F'} = M_F$ and $M_G = m_G = m_{G'}$ we have:

$$M_F \wedge_{\mathrm{PR}} m_{G'} = m_F \qquad M_F \vee_{\mathrm{PR}} m_{G'} = M_{G'}. \tag{$\diamond$}$$

This implies that the pair $(F_\star, F)$ and $(G, G^\star)$ and the pair $(F'_\star, F')$ and $(G', G'^\star)$ are both perspective. Indeed, looking at the first case $(F_\star, F)$ and $(G, G^\star)$, we want to show $F \wedge_{\mathbf{FW}} G = F_\star$ and $F \vee_{\mathbf{FW}} G = G^\star = G'$. For $F \wedge_{\mathbf{FW}} G = F_\star$, since F covers only $F_\star$ by definition of join-irreducible, it suffices to show $F_\star \le_{\mathbf{FW}} G$. Similarly, since G is only covered by $G^\star = G'$, to show $F \vee_{\mathbf{FW}} G = G'$, it suffices to show $F \le_{\mathbf{FW}} G'$. To show $F \le_{\mathbf{FW}} G'$ it suffices to observe that $m_F \le_{\mathrm{PR}} m_{G'}$ and $M_F \le_{\mathrm{PR}} M_{G'}$. Indeed, by ($\diamond$), we have $M_F \wedge_{\mathrm{PR}} m_{G'} = m_F$, implying $m_F \le_{\mathrm{PR}} m_{G'}$ and $M_F \vee_{\mathrm{PR}} m_{G'} = M_{G'}$ giving $M_F \le_{\mathrm{PR}} M_{G'}$ as desired. To show $F_\star \le_{\mathbf{FW}} G$ we follow a similar approach by proving that $m_{F_\star} \le_{\mathrm{PR}} m_G$ and $M_{F_\star} \le_{\mathrm{PR}} M_G$. Since $M_F \wedge_{\mathrm{PR}} m_G = M_F \wedge_{\mathrm{PR}} m_{G'} = m_F = m_{F_\star}$, therefore $m_{F_\star} \le_{\mathrm{PR}} m_G$. Also, since G and $F_\star$ are of codimension 0 we have $M_{F_\star} = m_{F_\star} \leq_{\mathrm{PR}} m_G = M_G$ as desired. Therefore $F_\star \leq_{\mathbf{FW}} G$ in the facial weak order.

The case $({F'}_\star, F')$ and $(G', G'^\star)$ is handled similarly.

Notice that the case where $\mathrm{codim}(F) = 0$ was handled in the proof above since F' is a join-irreducible element in the facial weak order with codimension 0. □

Combining this proposition with the fact that the lattice is self-dual, we get join-semidistributivity for free. In particular, we get that our lattice is semidistributive.

Theorem 4.3.27 *For a simplicial arrangement $\mathcal{A}$, the facial weak order is semidistributive.*

4.4 Topology of the facial weak order

In this section, we determine the homotopy type of intervals of the facial weak order; see Theorem 4.4.6. Before proving this theorem, some preliminary results on the topology of posets are given in § 4.4.1.

4.4.1 Poset topology

In this section, we recall some standard tools and definitions concerning simplicial complexes that we use in the proof of Theorem 4.4.6. The main result we need is Lemma 4.4.1.

An *abstract simplicial complex* Δ is a ground set E with a collection Δ of subsets of E (called *faces*) such that if $F \in \Delta$ and $G \subseteq F$ then $G \in \Delta$. The *dimension of a face* $F \in \Delta$ is given by $\dim(F) = |F| - 1$ and the *dimension of* Δ is the maximal dimension of all faces. An abstract simplicial complex of dimension d can be realized geometrically in $\mathbb{R}^{2d+1}$ by a union of simplices well-defined up to homeomorphism. We denote this geometrical realization by $\|\Delta\|$. The *deletion* $\mathrm{del}_\Delta(F)$ of F from Δ is the subcomplex of faces disjoint from F. The *link* $\mathrm{lk}_\Delta(F)$ of F is the subcomplex of faces G for which $F \cap G = \varnothing$ and $F \cup G$ is a face of Δ. The *join* $\Delta * \Delta'$ of two complexes with disjoint ground sets is the simplicial complex with faces $F \sqcup F'$ where $F \in \Delta,\ F' \in \Delta'$. The *cone* $\{v\} * \Delta$ is the join of Δ with a one-element complex. The *suspension* $\mathrm{susp}\,\Delta$ is the join of Δ with a discrete two-element complex. A fundamental homotopy equivalence connecting the deletion and link to the original complex is the following, which can be proved using the Carrier Lemma (*e.g.* (Walker, 1981, Lemma 2.1)).

Lemma 4.4.1 *Let F be a face of a simplicial complex Δ.*

(1) If $\mathrm{lk}_\Delta(F)$ is contractible, then Δ is homotopy equivalent to $\mathrm{del}_\Delta(F)$.

(2) If $\mathrm{del}_\Delta(F)$ is contractible, then Δ is homotopy equivalent to the suspension of $\mathrm{lk}_\Delta(F)$.

Given elements x, y of a poset P, the *open interval* (x, y) (resp. *closed interval* $[x, y]$) is the set of $z \in P$ such that $x < z < y$ (resp. $x \leq z \leq y$). We let $P_{<x}$ (resp. $P_{>x}$) denote the set of elements $y \in P$ such that $y < x$ (resp. $y > x$). The *order complex* $\Delta(P)$ of a poset P is the simplicial complex of chains $x_0 < \cdots < x_d$ of elements of P. The link of a face $x_0 < \cdots < x_d$ is isomorphic to the join of the order complexes of $P_{<x_0}, (x_0, x_1), \ldots, (x_{d-1}, x_d), P_{>x_d}$. Hence, the local topology of $\Delta(P)$ is completely determined by the topology of open intervals and principal order ideals and filters of P. In the remainder of this section, whenever we write about the topology of P, we mean the topology of its order complex.

In § 4.4.2, we explicitly determine the homotopy types of intervals of the facial weak order. To this end, we will use some consequences of Lemma 4.4.1.

Lemma 4.4.2 *Let P be a poset, and let $X \subseteq P$ such that $P_{<x}$ is contractible for all $x \in X$. Then P is homotopy equivalent to $P \setminus X$.*

Proof. Let $X = \{x_1, x_2, \ldots, x_n\}$ so that whenever $i \leq j$, we have $x_i \not< x_j$. We claim that P is homotopy equivalent to $P \setminus \{x_1, \ldots, x_i\}$ for any i. First observe that $\mathrm{lk}_{\Delta(P)}(x_1) = \Delta(P_{<x_1}) * \Delta(P_{>x_1})$ is contractible, so P is homotopy equivalent to $P \setminus \{x_1\}$. More generally, from the assumption on the ordering of elements of X, we have $\mathrm{lk}_{\Delta(P \setminus \{x_1,\ldots,x_{i-1}\})}(x_i) = \Delta(P_{<x_i}) * \Delta'$ for some simplicial complex Δ'. This is again contractible, so by induction, P is homotopy equivalent to $P \setminus \{x_1, \ldots, x_i\}$. Taking $i = n$, we have completed the proof. □

A *closure operator* (resp. *dual closure operator*) on a poset P is an idempotent, order preserving, increasing (resp. decreasing) function $f : P \to P$.

Lemma 4.4.3 (Corollary 10.12 (Björner, 1995)) *If $f : P \to P$ is a closure operator or a dual closure operator, then P is homotopy equivalent to $f(P)$.*

This lemma may be proved in many ways, *e.g.* by repeated application of Lemma 4.4.1 as we did for Lemma 4.4.2 or by application of Quillen's Fiber Lemma (Björner, 1995, Theorem 10.5).

4.4.2 Topology of intervals of the facial weak order

Let $\mathcal{A}$ be a real, central hyperplane arrangement with base region B. As usual, we orient the hyperplanes in $\mathcal{A}$ so that

$$B = \bigcap_{H \in \mathcal{A}} H^+.$$

Recall that the *poset of regions* $\mathrm{PR}(\mathcal{A}, B)$ is the set of regions with the partial order $R \leq_{\mathrm{PR}} R'$ if and only if $S(R) \subseteq S(R')$. With this ordering, B is the unique minimum element of the poset of regions. Given faces X, Y of $\mathcal{A}$, we say that X is *incident* to Y if $X \supseteq Y$.

P. H. Edelman and J. W. Walker determined the local topology of the poset of regions (Edelman & Walker, 1985). As this result will be used in the proof of Theorem 4.4.6, we state it here.

Theorem 4.4.4 ((Edelman & Walker, 1985)) *For each face $X \in \mathscr{F}_{\mathcal{A}}$, the set of regions incident to X is an interval $[R_1, R_2]_{\leq_{\mathrm{PR}}}$ of the poset of regions such that the open interval $(R_1, R_2)_{\leq_{\mathrm{PR}}}$ is homotopy equivalent to a sphere of dimension* $\operatorname{codim}(X) - 2$. *Every other interval is contractible.*

Recall that $\operatorname{span}(X)$ denotes the subspace spanned by a face X. The *poset of intersection subspaces*, or simply *intersection poset*, is the poset on the subspaces $L(\mathcal{A}) = \{\bigcap_{H \in \mathcal{I}} H \mid \mathcal{I} \subseteq \mathcal{A}\}$ ordered by reverse inclusion. As before this poset is a lattice when the vector space V is added as the bottom element and is called the *intersection lattice*. For X and Y in $L(\mathcal{A})$, the join is given by $X \vee_L Y = X \cap Y$ and the meet by $X \wedge_L Y = \bigcap_{X \cup Y \subseteq Z} Z$. For further information on the intersection lattice we refer the reader to P. Orlik and H. Terao's book (Orlik & Terao, 1992, Section 2). For a face X, let $\mathcal{A}^X$ denote the *restriction* of $\mathcal{A}$ to $\operatorname{span}(X)$ where

$$\mathcal{A}^X = \{H \cap \operatorname{span}(X) \mid H \in \mathcal{A} \setminus \mathcal{A}_X\}.$$

The set $\mathcal{A}^X$ is an arrangement of hyperplanes in the vector space spanned by X. For a covector X, we recall the map $\pi_X : \mathscr{F}_{\mathcal{A}} \to \mathscr{F}_{\mathcal{A}_X}$ where for any covector Y of $\mathcal{A}$, its image is the covector with $\pi_X(Y)(H) = Y(H)$ for $H \in \mathcal{A}_X$. Similarly, one can define a map $\iota^X : \mathscr{F}_{\mathcal{A}^X} \to \mathscr{F}_{\mathcal{A}}$ such that for a covector Y of $\mathcal{A}^X$ we have

$$\iota^X(Y)(H) = \begin{cases} Y(H) & \text{if } H \in \mathcal{A}^X \\ 0 & \text{if } H \in \mathcal{A}_X \end{cases}.$$

It is clear that ι^X is injective, so if Y is a face of $\mathcal{A}$ contained in $\mathrm{span}(X)$, we write $(\iota^X)^{-1}(Y)$ for the corresponding face of $\mathcal{A}^X$. To simplify notation, we define Y^X to be $(\iota^X)^{-1}(Y)$ in this case.

Lemma 4.4.5 *Let X, Y be covectors such that $X \leq_{\mathbf{FW}} Y$ in $\mathbf{FW}(\mathcal{A}, B)$, and let Z be a covector such that $\mathrm{span}(Z) = \mathrm{span}(X) \wedge_L \mathrm{span}(Y)$. Then the interval $[X, Y]$ of $\mathbf{FW}(\mathcal{A}, B)$ is isomorphic to the interval $[X^Z, Y^Z]$ of $\mathbf{FW}\Big(\mathcal{A}^Z, X^Z \circ (-Y^Z)\Big)$.*

Proof. For $H \in \mathcal{A}$ such that $X(H) = 0 = Y(H)$, any covector W in the interval $[X, Y]$ must satisfy $W(H) = 0$. Consequently, $\mathrm{span}(W) \subseteq \mathrm{span}(Z)$ for $W \in [X, Y]$, so the restriction W^Z is a covector of $\mathcal{A}^Z$. Moreover, the map $[X, Y] \to \mathcal{L}(\mathcal{A}^Z)$ is a bijection onto its image. Since $\mathrm{span}(X^Z) \wedge_L \mathrm{span}(Y^Z) = \mathrm{span}(Z)$ in the intersection lattice, the concatenation $X^Z \circ (-Y^Z)$ is a region of $\mathcal{A}^Z$. Moreover, for $H \in \mathcal{A}^Z$, we have $X^Z \circ (-Y^Z)(H) = X^Z(H)$ if $X^Z(H) \neq 0$, and $X^Z \circ (-Y^Z)(H) = +$ otherwise. Hence, $X^Z \leq_{\mathbf{FW}} Y^Z$ in $\mathbf{FW}\Big(\mathcal{A}^Z, X^Z \circ (-Y^Z)\Big)$, and if $W \in [X, Y]$, then $W^Z \in [X^Z, Y^Z]$. Conversely, every element of $[X^Z, Y^Z]$ is the restriction of some covector in $[X, Y]$. □

We are now ready to prove the main theorem. We make use of the fact that the proper part of the face lattice of a polyhedral cone of dimension d is homeomorphic to a sphere of dimension $d - 2$.

Theorem 4.4.6 *Let $\mathcal{A}$ be an arrangement with base region B. Let X, Y be covectors such that $X \leq_{\mathbf{FW}} Y$ and set $Z = X \cap Y$. If $X \leq_{\mathbf{FW}} Z \leq_{\mathbf{FW}} Y$ and $Z = X_{-Z} \cap Y$, then the order complex of the open interval (X, Y) in $\mathbf{FW}(\mathcal{A}, B)$ is homotopy equivalent to a sphere of dimension $\dim(X) + \dim(Y) - 2\dim(Z) - 2$. Every other interval is contractible.*

Proof. Let $X, Y \in \mathbf{FW}(\mathcal{A}, B)$ such that $X \leq_{\mathbf{FW}} Y$ and set $Z = X \cap Y$. Let Q be the open interval (X, Y) in the facial weak order. We determine the topology of Q.

If $Z = X$ then Q is an interval in the face lattice, so it is homeomorphic to a sphere of dimension $\dim(Y) - \dim(X) - 2$, as desired. Similarly, if $Z = Y$, then Q is homeomorphic to a sphere of dimension $\dim(X) - \dim(Y) - 2$. Hence, we may assume X, Y, Z are all distinct.

By Lemma 4.3.7, we may assume that $Z = 0$ since the poset $Q = (X, Y)$ is isomorphic to (X_Z, Y_Z). Hence, we write $-X$ for the covector X_{-Z}. By Lemma 4.4.5, we may assume that $\operatorname{span}(X) \wedge_L \operatorname{span}(Y) = V$ in the intersection lattice. In particular, $X \circ Y$ and $Y \circ X$ are regions. We will make these assumptions for most of the proof unless indicated otherwise.

Assume $Z \in Q$ and $Z = (-X) \cap Y$ both hold, and let Δ be the order complex of Q. We prove that $\operatorname{del}_\Delta(\{Z\})$ is contractible by induction on $\dim(Y)$.

Let $L_{>Y}$ denote the set of faces strictly less than the face Y in the face lattice, *i.e.*, $L_{>Y} = \{Z \mid Z >_L Y\} = \{Z \mid Z \subsetneq Y\}$. Applying the inductive hypothesis with Lemma 4.4.2, the poset $Q \setminus \{Z\}$ is homotopy equivalent to $Q \setminus L_{>Y}$. We note that this statement is vacuously true if $\dim(Y) = 1$. Set $P = Q \setminus L_{>Y}$. Define a map f on the closed interval $[X, Y]$ of the facial weak order, where $f(W) = W \circ Y$. This is well-defined by Lemma 4.2.11. We claim that f is a closure operator. It is clear that f is idempotent by properties of composition. Since $Z <_{\mathbf{FW}} Y$ in the facial weak order, every entry of Y is either 0 or $-$. Hence, f can only change some 0 entries of w to $-$, so it is order preserving and increasing. Since $W \circ Y \subseteq Y$ only if W is a face of Y, the operator f restricts to P. Lemma 4.4.3 implies that $P \simeq f(P)$.

Now define g on $[X, Y]$ where $g(W) = W \circ X$. This map is a dual closure operator. Assume that $W \in Q$ such that $g\big(f(W)\big) = X$. Then $f(W)$ must be a face of X. Since W is a face of $f(W)$, we deduce that W is a face of X. The set of faces of X intersected with $[X, Y]$ is an order ideal of $[X, Y]$ in the facial weak order. Since $(-X) \cap Y = 0$, the composite $X \circ Y$ is a region distinct from X. Then $X \circ Y \leq_{\mathbf{FW}} W \circ Y$, so $W \circ Y$ is not a face of X. This is a contradiction. Hence, g restricts to $f(P)$, and we conclude that $P \simeq g\Big(f(P)\Big)$.

Since X and Y have disjoint supports, the composite $Y \circ X$ is a region. Hence, the image of $g \circ f$ is the set of regions in Q. This set of regions has a maximum element, namely $Y \circ X$. Hence, it is contractible, as desired.

Since $\mathrm{del}_{\Delta}(\{Z\})$ is contractible, we conclude that Δ is homotopy equivalent to the suspension of $\mathrm{lk}_{\Delta}(\{Z\})$ by Lemma 4.4.1. By definition, $\mathrm{lk}_{\Delta}(\{Z\}) = \Delta((X,Z)) * \Delta((Z,Y))$. But $\Delta((X,Z))$ (resp. $\Delta((Z,Y))$) is the order complex of the proper part of the face lattice of the cone X (resp. Y). Hence, $\Delta((X,Z))$ is homeomorphic to $\mathbb{S}^{\dim(X)-\dim(Z)-2}$. Since $\mathbb{S}^p * \mathbb{S}^q \cong \mathbb{S}^{p+q+1}$ and $\mathrm{susp}(\mathbb{S}^p) \simeq \mathbb{S}^{p+1}$, we have

$$\begin{aligned}
\Delta &\simeq \mathrm{susp}\left(\mathrm{lk}_{\Delta}(\{Z\})\right)\\
&\simeq \mathrm{susp}\left(\Delta((X,Z)) * \Delta((Z,Y))\right)\\
&\simeq \mathrm{susp}\left(\mathbb{S}^{\dim(X)-\dim(Z)-2} * \mathbb{S}^{\dim(Y)-rk(Z)-2}\right)\\
&\simeq \mathbb{S}^{\dim(X)+\dim(Y)-2\dim(Z)-2}
\end{aligned}$$

Now assume that $Z \notin Q$. We prove that Q is contractible.

Since Z is not between X and Y, there exists $H \in \mathcal{A}$ such that $Z(H) = 0$ and either $X(H) = Y(H) = -$ or $X(H) = Y(H) = +$. Replacing B with $-B$, we may assume without loss of generality that $X(H) = Y(H) = -$ and $Z(H) = 0$. If W is any face of Y with $W \leq_{\mathbf{FW}} Y$, then $W(H) = -$. But $(W \cap X) \subseteq Z$, so $W \cap X$ is not between X and W. By induction, Q is homotopy equivalent to $Q \setminus L_{>Y}$. Let $P = Q \setminus L_{>Y}$. As before, we consider operators f and g on $[X,Y]$. These two operators again restrict to P, and $g\Big(f(P)\Big)$ is the subposet of regions in Q. If Y is not a region, then $Y \circ X$ is the unique maximum element of $g\Big(f(P)\Big)$. If X is not a region, then $X \circ Y$ is the unique minimum element of $g\Big(f(P)\Big)$. In either case, the interval $Q = (X, Y)$ is contractible. If both X and Y are regions, then Q is contractible by Theorem 4.4.4 since $g\big(f(P)\big)$ is an open interval of the poset of regions that is not facial.

Finally, assume that $Z \in Q$ but $Z \neq X_{-Z} \cap Y$. We prove that Q is contractible.

Assume X is not a region. Then $Y \cap (-X)$ is a proper face of Y, as otherwise there would exist a hyperplane $H \in \mathcal{A}$ containing both X and Y. Let

$$P = Q \setminus \{W \in L_{>Y} \mid \ (-X) \cap W \neq Z\}$$

and $P_{>Z} = \{W \in P \mid \ W > Z\}$. By induction, P is homotopy equivalent to Q. Then $P_{>Z}$ is contractible since $L_{>Y} \setminus \{Z\}$ is the proper part of the face lattice of the cone Y, and $P_{>Z}$ is the deletion of some face from this sphere.

Consequently, $P \simeq P \setminus \{Z\}$. We prove that $P \setminus \{Z\}$ is contractible by induction on $\dim(Y)$. We have already proved that $(X, Y') \setminus \{Z\}$ is contractible for $Y' \in L_{>Y} \cap P \setminus \{Z\}$. Hence, $P \setminus \{Z\} \simeq P \setminus L_{>Y}$. Set $P' = P \setminus L_{>Y}$. Using the operators f and g from before, we deduce that Q is homotopy equivalent to $g\big(f(P')\big)$. Since X is not a region, $g\big(f(P')\big)$ has a minimum element, namely $X \circ Y$. Hence, it is contractible.

If X is a region but Y is not a region, then a dual argument shows that Q is contractible. Hence, we may assume both X and Y are regions. Since this is the last remaining case, we deduce that for $W \in Q$ the interval (X, W) is not contractible if and only if W is an upper face of X. Hence, Q is homotopy equivalent to $L_{>X}$. This set of covectors has a maximum element in Q, namely $W = Z$. Hence, Q is contractible. □

4.4.3 Möbius function

Recall that the *Möbius function* of a poset P is the function $\mu : P \times P \to \mathbb{Z}$ defined inductively by

$$\mu(x,y) = \begin{cases} 1 & \text{if } x = y, \\ -\sum_{x \leq z < y} \mu(x,z) & \text{if } x < y, \\ 0 & \text{otherwise.} \end{cases}$$

For more information on the Möbius function we refer the reader to (Stanley, 2011).

We recall that the Möbius function can be restated using its homotopy type. In fact, $\mu(x,y)+1 = \sum(-1)^i \operatorname{rank} H_i(\Delta((x,y)))$ where $H_i(\Delta((x,y)))$ is the simplicial ith homology group and $\Delta((x,y))$ is the order complex for the open interval (x,y). The rank of the ith homology group is sometimes referred to as the *ith Betti number*.

Recall further that a contractible interval (x,y) has trivial homology (homotopy equivalent to a point). Thus $H_0(\Delta((x,y))) \cong \mathbb{Z}$ and $H_i(\Delta((x,y))) \cong \{0\}$ for all $i > 0$, *i.e.*, $\sum(-1)^i \operatorname{rank} H_i(\Delta((x,y))) = 1$. Therefore we have $\mu(x,y) = 0$. Additionally, recall that a sphere $\mathbb{S}^n$ has homology $H_0(\mathbb{S}^n) \cong \mathbb{Z}$, $H_n(\mathbb{S}^n) \cong \mathbb{Z}$ and $H_i(\mathbb{S}^n) \cong \{0\}$ for $0 < i < n$, *i.e.*, $\sum(-1)^i \operatorname{rank} H_i(\Delta((x,y))) = 1 + (-1)^n$. Therefore if our interval (x,y) is homotopy equivalent to $\mathbb{S}^n$ we have $\mu(x,y) = (-1)^n$. For more information on how the Möbius function relates to homology we refer the reader to the book by Stanley (Stanley, 2011), the book by Munkres (Munkres, 1984), or the chapter by Björner (Björner, 1995).

As a consequence to Theorem 4.4.6 we have the following corollary.

Corollary 4.4.7 *Let $\mathcal{A}$ be an arrangement with base region B. Let X, Y be* *covectors such that* $X \leq_{\mathbf{FW}} Y$ *and set* $Z = X \cap Y$.

$$\mu(X,Y) = \begin{cases} (-1)^{\dim(X)+\dim(Y)} & X \leq_{\mathbf{FW}} Z \leq_{\mathbf{FW}} Y \text{ and } Z = X_{-Z} \cap Y \\ 0 & \text{otherwise.} \end{cases}$$

CONCLUSION

In Chapter 1 we began by recalling the groups first introduced by Coxeter in (Coxeter, 1934). Recall that these groups, called Coxeter groups, are intricately related to reflection groups in the sense that each finite Coxeter group can be represented by a finite reflection group (and vice-versa). We then continued our study of Coxeter groups by defining the weak order, the order on the elements of a Coxeter group first defined by Björner in (Björner, 1984), which turns out to be a lattice for finite Coxeter groups. At the end of the first chapter we gave the definition of the Coxeter complex for a given finite Coxeter group W and showed that there is an isomorphism between the standard parabolic cosets of W and the faces of the W-permutahedron. In order to extend the weak order of W to the faces of the W-permutahedron, in Chapter 2 we introduced the facial weak order (see Definition 2.2.1) for Coxeter groups. The facial weak order turns out to be a lattice for finite Coxeter groups (see Theorem 2.2.19) extending the result that for finite Coxeter groups the weak order is a lattice.

We then generalized the results in Chapter 2 to hyperplane arrangements. Chapter 3 is dedicated to a survey on hyperplane arrangements and their associated faces. Since each finite Coxeter group has an associated hyperplane arrangement, called the Coxeter arrangement, we generalized the weak order to a poset on the regions of an arrangement. This poset of regions is different than the weak order in that it does not always produce a lattice like we would like. Fortunately, the poset of regions turns out to be a lattice for a large family of arrangements called simplicial arrangements (which include Coxeter arrangements). In order to replicate our extension of the weak order to the faces of the permutahedron, in

Chapter 4 we extended the poset of regions to an order on all faces of an arrangement. We called this extension the facial weak order for hyperplane arrangements and showed that the facial weak order is a lattice for simplicial arrangements (see Theorem 4.1 in Chapter 4), generalizing the result that the poset of regions is a lattice for simplicial arrangements.

These results are good starting points for research into the subject of the facial weak order. We end this conclusion by presenting further research that we will pursue in the coming years.

One of the first directions that we will pursue is to study in which case the facial weak order of oriented matroids is a lattice. A *tope* of an oriented matroid $(\mathcal{E}, \mathcal{L})$ is the generalization of a region of a hyperplane arrangement. With the set of topes, denoted $\mathcal{T}(\mathcal{L})$, the poset of regions is generalized to a tope poset by fixing a base tope and ordering the separation sets of the topes (from the base tope) by inclusion. A tope T is said to be *simplicial* if the interval $[\mathbf{0}, T]$ in the face lattice of an oriented matroid is isomorphic to a Boolean lattice. If every tope is simplicial then the oriented matroid is said to be simplicial. Similarly to simplicial hyperplane arrangements, it turns out that if the oriented matroid $(\mathcal{E}, \mathcal{L})$ is simplicial then the tope poset $(\mathcal{T}(\mathcal{L}), B, \leq_{\mathcal{T}})$ is a lattice for any choice of base tope B and if the tope poset $(\mathcal{T}(\mathcal{L}), B, \leq_{\mathcal{T}})$ is a lattice, then B must be simplicial (see (Björner et al., 1990, Theorems 6.3 and 6.5)). Due to this, one might conjecture that the facial weak order in simplicial oriented matroids is a lattice. This turns out to be the case if in addition to the oriented matroid being simplicial it is also assumed to be simple (there are no loops and no distinct parallel elements in $\mathcal{E}$). This can be proved using similar techniques as in the hyperplane arrangement case in Chapter 4.

In the case of hyperplane arrangements, we saw in Theorem 4.3.1 that if the poset of regions is a lattice then the facial weak order is a lattice. Additionally, when the arrangement is simplicial then Proposition 4.3.18 says the poset of regions is a sublattice of the facial weak order. This lead us to conjecture that when the facial weak order is a lattice the poset of regions will always be a lattice as well.

Conjecture 5.4.8 (Conjecture 4.4 in Chapter 4) *Let $\mathcal{A}$ be a (central) hyperplane arrangement with poset of regions* $\mathrm{PR}(\mathcal{A}, B)$*. Then* $\mathrm{PR}(\mathcal{A}, B)$ *is a lattice if and only if the facial weak order* $\mathbf{FW}(\mathcal{A}, B)$ *is a lattice.*

As another direction, in the cases where the facial weak order is a lattice, we will study the properties of the facial weak order. One such property that we believe the facial weak order to possess is congruence uniformity. A lattice is *congruence normal* if it can be obtained from the one-element lattice by a sequence of doublings of convex sets. As a stronger statement, a lattice is said to be *congruence uniform* if it can be obtained from the one-element lattice by a sequence of doublings of intervals, therefore giving us an algorithm for constructing the lattice in an efficient manner. It is known that the weak order on finite Coxeter groups is congruence uniform. This was first shown in the type A case by Caspard (see (Caspard, 2000)) and then for all finite Coxeter groups by Caspard, Le Conte de Poly-Barbut and Morvan (see (Caspard et al., 2004)). Since the facial weak order extends the weak order in finite Coxeter groups, we expect the congruence uniform property to hold in the facial weak order.

Conjecture 5.4.9 *The facial weak order on a finite Coxeter group is a congruence uniform lattice.*

This is the case for the small examples we have tried by hand. To prove this conjecture, it suffices to show that the facial weak order is congruence normal. This is because it is known that a lattice is congruence uniform if and only if it is both semidistributive and congruence normal (see (Day, 1994, Theorem 2.2)) and since we have already shown in Theorem 4.3.25 in Chapter 4 that the facial weak order is semidistributive for finite Coxeter groups.

In the case of simplicial hyperplane arrangements, it is not necessarily true that its poset of regions is a congruence uniform lattice. It turns out that the lattice of regions of a simplicial hyperplane arrangement is congruence uniform in only certain cases. We direct the reader to (Reading, 2003, §8) for more details on when a lattice of regions for a simplicial hyperplane arrangement is congruence uniform. For our purposes, we believe that the congruence uniform property should extend into the facial weak order whenever it holds for the underlying poset of regions.

Conjecture 5.4.10 *Let $\mathcal{A}$ be a simplicial hyperplane arrangement whose lattice of regions $(\mathscr{R}, B, \leq_{\mathcal{A}})$ is congruence uniform. Then the facial weak order $\mathbf{FW}(\mathcal{A}, B)$ is congruence uniform.*

If true, this conjecture would provide a lot of information on the congruences of the facial weak order. For instance, the fact that if a lattice $\mathscr{L}$ is congruence uniform, implies the join-irreducible elements of $\mathscr{L}$ are in bijection with the join-irreducible elements of the lattice of congruences, $\mathbf{con}(\mathscr{L})$, of $\mathscr{L}$ (see (Day, 1994, p. 400)). Since we gave a description of the join-irreducible elements of $\mathbf{FW}(\mathcal{A}, B)$ in Proposition 4.28 in Chapter 4 this would give us a deeper understanding of the join-irreducible elements of $\mathbf{con}(\mathbf{FW}(\mathcal{A}, B))$.

Finally, as a long term project, we would try to replicate the facial weak order on arbitrary polytopes and not just zonotopes. This would involve finding a proper extension of the facial weak order for polytopes. One approach would involve the inner primal cones as we did in both Coxeter groups and hyperplane arrangement cases through the root inversion sets. If a facial weak order can be defined in this way for polytopes, it would be interesting to know what properties a polytope must possess for the facial weak order to be a lattice. Finding such a definition of the facial weak order on polytopes is already not so evident in 2-dimensional Euclidean space. As an example take the equilateral triangle and start from any vertex v. Unlike the case with a $2n$-polygon which has a unique face (a vertex) farthest from a staring vertex, for the equilateral triangle there are three faces that are "equally" far from v (the two other vertices and the edge between them) depending on our choice of definition for facial weak order. Even with a generic linear functional (*i.e.*, no two vertices lie at the same height) orienting the skeleton of the polytope, it is unclear which polytopes admit such an orientation for which their skeleton is the Hasse diagram. Finally, once a proper and natural definition for the facial weak order is found on polytopes, the question of, for which polytopes is the facial weak order a lattice, will be opened.

APPENDIX A

ORDER THEORY

In this appendix we survey the topic of order theory. We start by recalling partial orders on sets (posets) in § A.1 and give a presentation of posets using diagrams. Then in § A.2 we introduce the notions of joins and meets: a generalization of greatest common divisor and least common multiple. Lattices, a special family of posets where every two elements have a join and a meet, are surveyed in § A.3. Finally, we describe maps between posets and lattices in § A.4. For a more introductory background with more example on posets, the reader is referred to the book "Enumerative Combinatorics: Volume 1" by Stanley (Stanley, 2011). For further background on lattices, the interested reader is referred to the book "Lattice theory" by Birkhoff (Birkhoff, 1967) or the book "General lattice theory" by Grätzer (Grätzer, 1986).

A.1 Partial orders

In this section we recall the basic definitions of posets.

Let $(P, \preccurlyeq)$ be a partially ordered set (or *poset* for short). Recall that a and b in P are said to be *incomparable* if there is not a relation between a and b, *i.e.*, neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, otherwise they are said to be *comparable*. A poset in which every two elements are comparable is known as a *totally ordered set* and the order $\preccurlyeq$ is known as a *total order*. If P is finite we call the poset *finite*, else we call it *infinite*.

Example A.1.1 Let $\mathbb{N}^\star = \{1, 2, 3, \ldots\}$ be the set of natural numbers and let $\leq$ be the standard less than or equal to sign. Then $\leq$ is a total order and the infinite poset $(\mathbb{N}^\star, \leq)$ is a totally ordered set since there is a relation between every two elements. As examples of the order $\leq$ we have the following relations: $3 \leq 5$, $9 \leq 10$, $2 \leq 2$, $1 \leq 9{,}001$, etc.

Example A.1.2 Consider $\mathbb{N}^\star$ and let the partial order $|$ be such that for $a, b \in \mathbb{N}^\star$, then $a \mid b$ if and only if a divides b, *i.e.*, $\frac{b}{a} \in \mathbb{N}^\star$. In this poset we have $2 \mid 4$ since $\frac{4}{2} = 2 \in \mathbb{N}^\star$. Similarly, $2 \mid 42$ since $\frac{42}{2} = 21 \in \mathbb{N}^\star$.

Unlike our previous example, there are some numbers which are incomparable. For example, neither $2 \mid 3$ nor $3 \mid 2$ are relations in this poset. This is due to the fact that $\frac{2}{3}$ and $\frac{3}{2}$ are not integers. Therefore, 3 and 2 are incomparable.

We let $(\mathbb{N}^\star, |)$ denote this poset.

Example A.1.3 Let $[n] = \{1, 2, 3, \ldots, n\}$ and let $\mathbf{2}^n$ denote all subsets of $[n]$, *i.e.*, $\mathbf{2}^n = \{\mathcal{N} \mid \mathcal{N} \subseteq [n]\}$. Ordering the elements by inclusion gives the poset $(\mathbf{2}^n, \subseteq)$, called the *Boolean poset*. As examples of relations in $(\mathbf{2}^2, \subseteq)$, we have $\{1\} \subseteq \{1, 2\} = [2]$ and $\varnothing \subseteq \{2\}$. As with our last example, the Boolean poset has elements which are incomparable. For example, $\{1\}$ and $\{2\}$ are incomparable since neither is a subset of the other.

An element b in P *covers* an element a in P (or a is *covered by* b), denoted $a \prec\!\cdot\; b$, if $a \prec b$ and if there does not exist any $c \in P$ such that $a \prec c \prec b$. Recall that

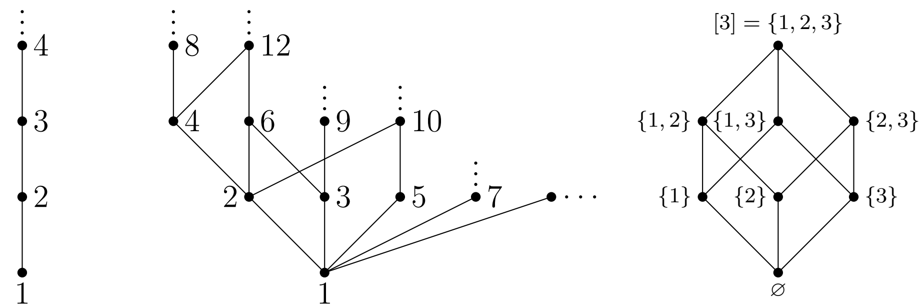


Figure A.1: The Hasse diagrams for $(\mathbb{N}^\star, \leq)$, $(\mathbb{N}^\star, |)$, and $(\mathbf{2}^3, \subseteq)$ respectively.

the *Hasse diagram*[1] of $(P, \preccurlyeq)$ is a graph with vertex set P and an upward edge from a to b whenever $a \prec\!\cdot\; b$.

The Hasse diagrams for $(\mathbb{N}^\star, \leq)$, $(\mathbb{N}^\star, |)$ and $(\mathbf{2}^n, \subseteq)$ are given in Figure A.1 from left to right respectively.

The poset $(\mathbb{N}^\star, |)$ has an interesting property in that for any two elements in $\mathbb{N}^\star$ there is a greatest common divisor and a least common multiple. We generalize this property to arbitrary posets in the next section.

A.2 Joins and Meets

In the poset $(\mathbb{N}^\star, |)$ every two elements have a greatest common divisor (gcd) and a least common multiple (lcm). In this section we extend this property to posets in general.

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[1]According to Birkhoff (Birkhoff, 1948) the term "Hasse diagram" was named after Hasse due to his effective use of these diagrams in the 1920s in field theory and number theory (see (Lemmermeyer & Roquette, 2012, § 3.18, § 5.2 and § 6.5) for examples).The first known instance of a Hasse diagram can be traced back to at least 1895 by Vogt in (Vogt, 1895, p. 91) where he uses these diagrams to show which groups are subgroups of other groups for the symmetric group S_4.Additionally, these diagrams also have a rich history in genealogy.

Given a poset $(P, \preccurlyeq)$ and two elements $a, b \in P$ then $c \in P$ is an *upper bound* of a and b if $a \preccurlyeq c$ and $b \preccurlyeq c$. Similarly, a *lower bound* of a and b is an element $d \in P$ such that $d \preccurlyeq a$ and $d \preccurlyeq b$. The *join* (or *least upper bound*) of a and b (if it exists) is the unique upper bound $z \in P$ such for every upper bound c of a and b we have $z \preccurlyeq c$. Similarly, the *meet* (or *greatest lower bound*) of a and b (if it exists) is the unique lower bound $y \in P$ such that for every lower bound d of a and b we have $d \preccurlyeq y$. If they exist, we denote the join of a and b as $a \vee b$ and the meet of a and b as $a \wedge b$.

Examples A.2.1 In the examples we've been using, every two elements have a meet and a join. For the poset $(\mathbb{N}^\star, |)$ the join is the lcm and the meet is the gcd.

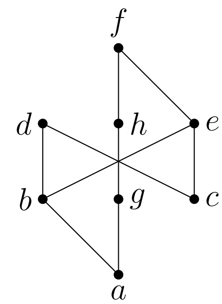
For the poset $(\mathbb{N}^\star, \leq)$ the join is the greater of the two numbers and the meet is the lesser of the two numbers. In fact, for all total orders, since we always have either $a \preccurlyeq b$ or $b \preccurlyeq a$, then every two elements must have a join which is the greater of the two, and a meet, which is the lesser of the two.

For the poset $(\mathbf{2}^n, \subseteq)$ the join is given by the union of the two sets and the meet is given by the intersection of the two sets. As an example, in $(\mathbf{2}^6, \subseteq)$ we have

$$\{1,4\} \vee \{2,4,5\} = \{1,2,4,5\} \quad \text{and} \quad \{1,4\} \wedge \{2,4,5\} = \{4\}\,.$$

Although in our examples we always have a join and a meet, this is not necessarily always the case for arbitrary posets.

Example A.2.2 Let $(P, \preccurlyeq)$ be the poset with the following Hasse diagram:



From the Hasse diagram we deduce that $a \prec\cdot\, b$, $a \preccurlyeq d$, $c \preccurlyeq e$, etc. We can also use the Hasse diagram to help find the meets and joins of particular elements. For example, the join of e and g is f, *i.e.*, $e \vee g = f$. This can be observed from the fact that $e \preccurlyeq f$, $g \preccurlyeq f$ and there does not exist any other upper bound of e and g. As another example, the meet of e and g is a, *i.e.*, $e \wedge g = a$.

Although e and g have both a meet and a join, it can be observed that not every two elements have a meet and join. For example the join $b \vee c$ does not exist since the set of upper bounds $\{d, e, f\}$ of both b and c does not contain an element which is weakly below every upper bound. Additionally, the meet of b and c, $b \wedge c$, does not exist either since there are no lower bounds of both b and c.

Having a meet and a join for every pair of elements is a special property which many posets have called the lattice property.

A.3 Lattices

In this section we survey lattices: posets where every pair of elements have a meet and a join.

Let $(L, \preccurlyeq)$ be a poset. If every two elements in L have a join then the poset $(L, \preccurlyeq)$ is called a *join-semilattice*. Similarly, if every two elements in L have a meet, then $(L, \preccurlyeq)$ is said to be a *meet-semilattice*. If $(L, \preccurlyeq)$ is both a join-semilattice and a meet-semilattice, then $(L, \preccurlyeq)$ is called a *lattice*. If L is finite then we call the lattice *finite*, else *infinite*.

For a finite lattice, since every two elements have a join, there is always some element that is greater than every other element. This element is known as the *top element* and is denoted by $\hat{1}$. Note that this is not the case with infinite lattices as can be observed with $(\mathbb{N}^\star, \leq)$. Similarly, for a finite lattice, there is always a unique *bottom element*, denoted by $\hat{0}$, since every two elements have a meet. The elements which cover the bottom element are called *atoms*. In other words, if a is an atom then $\hat{0} \prec\!\cdot\; a$. Similarly, *coatoms* are elements which are covered by the top element $\hat{1}$.

Example A.3.1 The three posets $(\mathbb{N}^\star, \leq)$, $(\mathbb{N}^\star, |)$ and $(\mathbf{2}^n, \subseteq)$ are all lattices since every pair of elements has a meet and a join. On the other hand, the poset in Example A.2.2 is not a meet-semilattice (not every two elements have a meet) nor is it a join-semilattice (not every two elements have a join) and therefore it is not a lattice. Since the Boolean poset $(\mathbf{2}^n, \subseteq)$ is a finite lattice, it has a unique top element given by $\hat{1} = [n]$ and a unique bottom element given by $\hat{0} = \varnothing$. The Boolean poset is commonly referred to as the *Boolean lattice* since it is a lattice.

Given a poset $(P, \preccurlyeq)$ it might not always be easy to verify if every pair of elements has a meet and a join. An alternative method is to show that $(P, \preccurlyeq)$ is isomorphic to a lattice. We survey maps between posets in the next section.

A.4 Poset Isomorphisms

In this section, we survey the notion of maps and isomorphisms between posets and lattices.

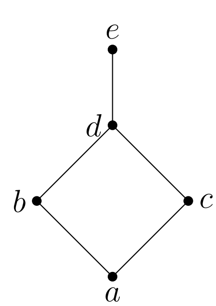
Let $(P, \preccurlyeq_P)$ and $(Q, \preccurlyeq_Q)$ be two posets. A function $\theta : P \to Q$ is an *order-*

preserving map if for $p, p' \in P$ then $p \preccurlyeq_P p'$ implies $\theta(p) \preccurlyeq_Q \theta(p')$. The function θ is *order-reversing* if $p \preccurlyeq_P p'$ implies $\theta(p') \preccurlyeq_Q \theta(p)$. If the order-preserving map θ is a bijection whose inverse is order-preserving ($p \preccurlyeq_P p'$ if and only if $\theta(p) \preccurlyeq_Q \theta(p')$) then θ is called a *poset isomorphism* and $(P, \preccurlyeq_P)$ and $(Q, \preccurlyeq_Q)$ are said to be *isomorphic*.

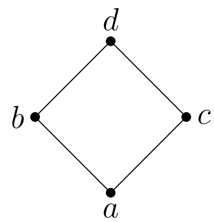
Given a poset $(P, \preccurlyeq)$, an *induced subposet* (or *subposet* for short) $(P', \preccurlyeq)$ of $(P, \preccurlyeq)$ is the poset restricted to the elements $P' \subseteq P$ and where for $p, p' \in P'$, $p \preccurlyeq p'$ in $(P', \preccurlyeq)$ if and only if $p \preccurlyeq p'$ in $(P, \preccurlyeq)$. If $(L, \preccurlyeq)$ is a lattice, then a *sublattice* of $(L, \preccurlyeq)$ is a subposet $(L', \preccurlyeq)$ such that, if $a, b \in L'$, then $a \wedge b$ and $a \vee b$ are also in L'. It should be noted that a subposet $(L', \preccurlyeq)$ of $(L, \preccurlyeq)$ can be a lattice without being a sublattice of $(L, \preccurlyeq)$. A useful subposet of any poset is an interval of the poset. An *interval* in a poset $(P, \preccurlyeq)$ is a set $[p, p']$ with $p, p' \in P$ where $[p, p'] = \{x \in P \mid p \preccurlyeq x \preccurlyeq p'\}$. Given a lattice $\mathscr{L} = (L, \preccurlyeq)$, then it can be verified that any interval $[x, y]$ in $\mathscr{L}$ is a sublattice $([x, y], \preccurlyeq)$ of $\mathscr{L}$.

Example A.4.1 Let $(\mathbf{2}^n, \subseteq)$ be the Boolean lattice of $[n]$. A sublattice of $(\mathbf{2}^n, \subseteq)$ is the Boolean lattice $(\mathbf{2}^m, \subseteq)$ where $m \leq n$. In fact, $(\mathbf{2}^m, \subseteq)$ is the interval $[\varnothing, [m]]$ in the lattice $(\mathbf{2}^n, \subseteq)$. There is an injective map from $(\mathbf{2}^m, \subseteq)$ to $(\mathbf{2}^n, \subseteq)$ which sends a set $\mathcal{M} \in \mathbf{2}^m$ to the set $\mathcal{M} \in \mathbf{2}^n$. It is easily verified that this map is order-preserving. Similarly, there is a projective map from $(\mathbf{2}^n, \subseteq)$ to $(\mathbf{2}^m, \subseteq)$ which sends $\mathcal{N} \in \mathbf{2}^n$ to $(\mathcal{N} \backslash ([n] \backslash [m])) \in \mathbf{2}^m$. This map is also order-preserving.

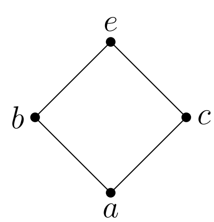
Example A.4.2 Let $(P, \preccurlyeq)$ be the lattice with the following Hasse diagram:



Let $P' = \{a, b, c, d\}$ and $P'' = \{a, b, c, e\}$ be two subsets of P. Then the induced subposet $(P', \preccurlyeq)$ of $(P, \preccurlyeq)$ has the following Hasse diagram



and the induced subposet $(P'', \preccurlyeq)$ of $(P, \preccurlyeq)$ has the following Hasse diagram



Although both of these subposets are lattices, only $(P', \preccurlyeq)$ is a sublattice of $(P, \preccurlyeq)$. The subposet $(P'', \preccurlyeq)$ is not a sublattice since $b \vee_P c = d \notin P''$. An easy way to verify that $(P', \preccurlyeq)$ is a sublattice of $(P, \preccurlyeq)$ is to notice that P' is in fact the interval $[a, d]$ in the lattice $(P, \preccurlyeq)$, *i.e.*, $(P', \preccurlyeq) = ([a, d], \preccurlyeq)$.

The two subposets $(P', \preccurlyeq)$ and $(P'', \preccurlyeq)$ are isomorphic to one another through the map $\theta : (P', \preccurlyeq) \to (P'', \preccurlyeq)$ where $a \mapsto a$, $b \mapsto b$, $c \mapsto c$ and $d \mapsto e$.

The *dual* of a poset is the poset $(P, \preccurlyeq_P^\star)$ where $p \preccurlyeq_P^\star p'$ if and only if $p' \preccurlyeq_P p$. The Hasse diagram for the dual poset can be thought of as taking the Hasse diagram of the original poset and flipping it upside down. If $(P, \preccurlyeq_P)$ is isomorphic to $(P, \preccurlyeq_P^\star)$ then we say that $(P, \preccurlyeq_P)$ is *self-dual* . We have the following theorem, see for instance (Birkhoff, 1967).

Theorem A.4.3 *Let $(P, \preccurlyeq)$ be a self-dual poset with $a, b \in P$. Then*

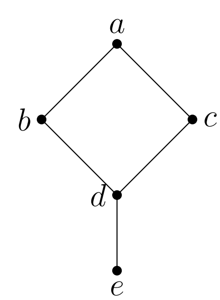
$$a \wedge b \text{ exists if and only if } a \vee b \text{ exists.}$$

Example A.4.4 The Boolean lattice $(\mathbf{2}^n, \subseteq)$ is self-dual. To observe this, let θ be the following map

$$\theta : (\mathbf{2}^n, \subseteq) \to (\mathbf{2}^n, \supseteq) \text{ such that } \mathcal{N} \mapsto [n] \backslash \mathcal{N}.$$

The poset $(\mathbf{2}^n, \supseteq)$ is dual to $(\mathbf{2}^n, \subseteq)$ since $\mathcal{N} \subseteq \mathcal{M}$ implies $\mathcal{M} \supseteq \mathcal{N}$. Furthermore, this map is an isomorphism since it preserves the order, is a bijection, and the reverse map ($\mathcal{N} \supseteq \mathcal{M} \mapsto [n] \backslash \mathcal{N} \subseteq [n] \backslash \mathcal{M}$) is order-preserving as well.

Example A.4.5 Using the lattice $(P, \preccurlyeq)$ as in Example A.4.2, we can observe that the lattice $(P, \preccurlyeq)$ is not self-dual. This is because there is no order-preserving bijection from $(P, \preccurlyeq)$ to its dual, whose Hasse diagram is the following



For lattices, we can use joins and meets in order to verify whether an isomorphism is present. A *join-homomorphism* is a map θ from a lattice $(L, \preccurlyeq_L)$ to a lattice $(M, \preccurlyeq_M)$ such that $\theta(l \vee_L l') = \theta(l) \vee_M \theta(l')$ for all $l, l' \in L$. Similarly, a *meet-homomorphism* is a map θ such that $\theta(l \wedge_L l') = \theta(l) \wedge_M \theta(l')$ for all $l, l' \in L$. A *homomorphism* is a map $\theta : (L, \preccurlyeq_L) \to (M, \preccurlyeq_M)$ such that θ is both a join-homomorphism and a meet-homomorphism. Then a *lattice isomorphism* is a lattice homomorphism which is bijective. Lattice isomorphisms are equivalent to poset isomorphisms where our two posets are lattices, see (Birkhoff, 1967, Lemmas II.3.1 and II.3.2)

Theorem A.4.6 *Given two lattices* $(L, \preccurlyeq_L)$ *and* $(M, \preccurlyeq_M)$ *and a map*

$$\theta : (L, \preccurlyeq_L) \rightarrow (M, \preccurlyeq_M)$$

between the two, then θ *is a lattice isomorphism if and only if it is a poset isomorphism.*

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